Separable Equation

We are now going to start looking at nonlinear first order differential equations. The first type of nonlinear first order differential equations that we will look at is separable differential equations.

A separable differential equation is any differential equation that we can write in the following form.

$$N(y)\frac{dy}{dx} = M(x)$$

Note that in order for a differential equation to be separable all the y's in the differential equation must be multiplied by the derivative and all the x's in the differential equation must be on the other side of the equal sign.

To solve this differential equation we first integrate both sides with respect to *x* to get,

$$\int N(y)\frac{dy}{dx}dx = \int M(x)dx$$

Now we integrate both sides of this to get,

$$\int N(y)dy = \int M(x)dx$$

Example 1 Solve the following differential equation and determine the interval of validity for the solution.

$$\frac{dy}{dx} = 6y^2x \qquad y(1) = \frac{1}{25}$$

Solution

$$y^{-2}dy = 6x\,dx$$

$$\int y^{-2}dy = \int 6x\,dx$$

$$-\frac{1}{y} = 3x^2 + c$$

So, we now have an implicit solution. This solution is easy enough to get an explicit solution, however before getting that it is usually easier to find the value of the constant at this point. So apply the initial condition and find the value of *c*.

$$-\frac{1}{1/25} = 3(1)^2 + c \qquad c = -28$$

Plug this into the general solution and then solve to get an explicit solution.

$$-\frac{1}{y} = 3x^2 - 28$$

$$y(x) = \frac{1}{28 - 3x^2}$$

Recall that there are two conditions that define an interval of validity. First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition, $x = 1$ in this case.

So, for our case we've got to avoid two values of x. Namely, $x \neq \pm\sqrt{\frac{28}{3}} \approx \pm 3.05505$ since these will give us division by zero. This gives us three possible intervals of validity.

$$-\infty < x < -\sqrt{\frac{28}{3}} \qquad -\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}} \qquad \sqrt{\frac{28}{3}} < x < \infty$$

Example 2 Solve the following IVP and find the interval of validity for the solution.

$$y' = \frac{3x^2 + 4x - 4}{2y - 4} \qquad y(1) = 3$$

Solution

This differential equation is clearly separable, so let's put it in the proper form and then integrate both sides.

$$(2y-4)\,dy = (3x^2 + 4x - 4)\,dx$$
$$\int (2y-4)\,dy = \int (3x^2 + 4x - 4)\,dx$$
$$y^2 - 4y = x^3 + 2x^2 - 4x + c$$

We now have our implicit solution, so as with the first example let's apply the initial condition at this point to determine the value of c.

$$(3)^2 - 4(3) = (1)^3 + 2(1)^2 - 4(1) + c \qquad c = -2$$

The implicit solution is then

$$y^2 - 4y = x^3 + 2x^2 - 4x - 2$$

We now need to find the explicit solution. This is actually easier than it might look and you already know how to do it. First, we need to rewrite the solution a little

$$y^2 - 4y - (x^3 + 2x^2 - 4x - 2) = 0$$

To solve this all we need to recognize is that this is quadratic in y and so we can use the quadratic formula to solve it. However, unlike quadratics you are used to, at least some of the "constants" will not actually be constant but will in fact involve x's.

So, upon using the quadratic formula on this we get.

$$y(x) = \frac{4 \pm \sqrt{16 - 4(1)\left(-(x^3 + 2x^2 - 4x - 2)\right)}}{2}$$
$$= \frac{4 \pm \sqrt{16 + 4(x^3 + 2x^2 - 4x - 2)}}{2}$$

Next, notice that we can factor a 4 out from under the square root (it will come out as a 2...) and then simplify a little.

$$y(x) = \frac{4 \pm 2\sqrt{4 + (x^3 + 2x^2 - 4x - 2)}}{2}$$
$$= 2 \pm \sqrt{x^3 + 2x^2 - 4x + 2}$$

We are almost there. Notice that we've actually got two solutions here (the " $\pm$ ") and we only want a single solution. In fact, only one of the signs can be correct. So, to figure out which one is correct we can reapply the initial condition to this. Only one of the signs will give the correct value so we can use this to figure out which one of the signs is correct. Plugging $x = 1$ into the solution gives.

$$3 = y(1) = 2 \pm \sqrt{1 + 2 - 4 + 2} = 2 \pm 1 = 3, 1$$

In this case it looks like the "+" is the correct sign for our solution. Note that it is completely possible that the "–" could be the solution (*i.e.* using an initial condition of $y(1) = 1$) so don't always expect it to be one or the other.

The explicit solution for our differential equation is.

$$y(x) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}$$

To finish the example out we need to determine the interval of validity for the solution. If we were to put a large negative value of x in the solution we would end up with complex values in our solution and we want to avoid complex numbers in our solutions here. So, we will need to determine which values of x will give real solutions. To do this we will need to solve the following inequality.

$$x^3 + 2x^2 - 4x + 2 \geq 0$$

Therefore, the interval of validity of the solution is $x \geq -3.36523$.

Example 3 Solve the following IVP and find the interval of validity of the solution.

$$y' = \frac{xy^3}{\sqrt{1 + x^2}} \qquad y(0) = -1$$

Solution

$$y^{-3}dy = x(1 + x^2)^{-\frac{1}{2}}\,dx$$
$$\int y^{-3}dy = \int x(1 + x^2)^{-\frac{1}{2}}\,dx$$
$$-\frac{1}{2y^2} = \sqrt{1 + x^2} + c$$

Apply the initial condition to get the value of *c*.

$$-\frac{1}{2} = \sqrt{1} + c \qquad\qquad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now let's solve for *y(x)*.

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

$$y(x) = \pm\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Reapplying the initial condition shows us that the "–" is the correct sign. The explicit solution is then,

$$y(x) = -\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Let's get the interval of validity. That's easier than it might look for this problem. First, since $1 + x^2 \geq 0$ the "inner" root will not be a problem. Therefore, all we need to worry about is division by zero and negatives under the "outer" root. We can take care of both by requiring

$$3 - 2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4\left(1+x^2\right)$$

$$\frac{9}{4} > 1 + x^2$$

$$\frac{5}{4} > x^2$$

Note that we were able to square both sides of the inequality because both sides of the inequality are guaranteed to be positive in this case. Finally solving for *x* we see that the only possible range of *x*'s that will not give division by zero or square roots of negative numbers will be,

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

Example 4 Solve the following IVP and find the interval of validity of the solution.

$$y' = \mathbf{e}^{-y}(2x-4) \qquad y(5) = 0$$

Solution

This differential equation is easy enough to separate, so let's do that and then integrate both sides.

$$\mathbf{e}^{y}\,dy = (2x-4)\,dx$$

$$\int \mathbf{e}^{y}\,dy = \int (2x-4)\,dx$$

$$\mathbf{e}^{y} = x^2 - 4x + c$$

Applying the initial condition gives

$$1 = 25 - 20 + c \qquad c = -4$$

This then gives an implicit solution of.

$$\mathbf{e}^{y} = x^2 - 4x - 4$$

We can easily find the explicit solution to this differential equation by simply taking the natural log of both sides.

$$y(x) = \ln\left(x^2 - 4x - 4\right)$$

Finding the interval of validity is the last step that we need to take. Recall that we can't plug negative values or zero into a logarithm, so we need to solve the following inequality

$$x^2 - 4x - 4 > 0$$

The quadratic will be zero at the two points $x = 2 \pm 2\sqrt{2}$

So, possible intervals of validity are

$$-\infty < x < 2 - 2\sqrt{2}$$

$$2 + 2\sqrt{2} < x < \infty$$

Example 5 Solve the following IVP and find the interval of validity for the solution.

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} \qquad r(1) = 2$$

Solution

This is actually a fairly simple differential equation to solve. We're doing this one mostly because of the interval of validity.

So, get things separated out and then integrate.

$$\frac{1}{r^2}dr = \frac{1}{\theta}d\theta$$

$$\int \frac{1}{r^2}dr = \int \frac{1}{\theta}d\theta$$

$$-\frac{1}{r} = \ln|\theta| + c$$

Now, apply the initial condition to find *c*.

$$-\frac{1}{2} = \ln(1) + c \qquad c = -\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

Solving for *r* gets us our explicit solution.

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

Now, there are two problems for our solution here. First, we need to avoid $\theta = 0$ because of the natural log. Notice that because of the absolute value on the θ we don't need to worry about θ being negative. We will also need to avoid division by zero. In other words, we need to avoid the following points.

$$\frac{1}{2} - \ln|\theta| = 0$$

$$\ln|\theta| = \frac{1}{2} \qquad \text{exponentiate both sides}$$

$$|\theta| = \mathbf{e}^{\frac{1}{2}}$$

$$\theta = \pm\sqrt{\mathbf{e}}$$

So, these three points break the number line up into four portions, each of which could be an interval of validity.

$$-\infty < \theta < -\sqrt{\mathbf{e}}$$
$$-\sqrt{\mathbf{e}} < \theta < 0$$
$$0 < \theta < \sqrt{\mathbf{e}}$$
$$\sqrt{\mathbf{e}} < \theta < \infty$$

The interval that will be the actual interval of validity is the one that contains $\theta = 1$. Therefore, the interval of validity is $0 < \theta < \sqrt{\mathbf{e}}$.

Example 6 Solve the following IVP.

$$\frac{dy}{dt} = \mathbf{e}^{y-t}\sec(y)\left(1+t^2\right) \qquad y(0) = 0$$

Solution

This problem will require a little work to get it separated and in a form that we can integrate, so let's do that first.

$$\frac{dy}{dt} = \frac{\mathbf{e}^{y}\mathbf{e}^{-t}}{\cos(y)}\left(1+t^2\right)$$
$$\mathbf{e}^{-y}\cos(y)\,dy = \mathbf{e}^{-t}\left(1+t^2\right)dt$$

Now, with a little integration by parts on both sides we can get an implicit solution.

$$\int \mathbf{e}^{-y}\cos(y)\,dy = \int \mathbf{e}^{-t}\left(1+t^2\right)dt$$
$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y) - \cos(y)\right) = -\mathbf{e}^{-t}\left(t^2 + 2t + 3\right) + c$$

Applying the initial condition gives.

$$\frac{1}{2}(-1) = -(3) + c \qquad c = \frac{5}{2}$$

Therefore, the implicit solution is.

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y) - \cos(y)\right) = -\mathbf{e}^{-t}\left(t^2 + 2t + 3\right) + \frac{5}{2}$$

It is not possible to find an explicit solution for this problem and so we will have to leave the solution in its implicit form. Finding intervals of validity from implicit solutions can often be very difficult so we will also not bother with that for this problem.

As this last example showed it is not always possible to find explicit solutions so be on the lookout for those cases.