Exact Equation

A differential equation of type

$$P(x, y)dx + Q(x, y)dy = 0$$

is called an exact differential equation if there exists a function of two variables $u(x, y)$ with continuous partial derivatives such that

$$du(x, y) = P(x, y)dx + Q(x, y)dy.$$

The general solution of an exact equation is given by

$$u(x, y) = C,$$

where C is an arbitrary constant.

Test for Exactness

Let functions $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives in a certain domain D. The differential equation $P(x, y)dx + Q(x, y)dy = 0$ is an exact equation if and only if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Algorithm for Solving an Exact Differential Equation

1 First it's necessary to make sure that the differential equation is exact using the test for exactness:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

2 Then we write the system of two differential equations that define the function $u(x, y)$:

$$\begin{cases} \frac{\partial u}{\partial x} = P(x, y) \\ \frac{\partial u}{\partial y} = Q(x, y) \end{cases}.$$

3 Integrate the first equation over the variable x. Instead of the constant C, we write an unknown function of y :

$$u(x, y) = \int P(x, y)dx + \varphi(y).$$

4 Differentiating with respect to y, we substitute the function $u(x, y)$into the second equation:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int P(x, y)dx + \varphi(y)\right] = Q(x, y).$$

From here we get expression for the derivative of the unknown function $\varphi(y)$:

$$\varphi'(y) = Q(x, y) - \frac{\partial}{\partial y}\left(\int P(x, y)dx\right).$$

5 By integrating the last expression, we find the function $\varphi(y)$ and, hence, the function $u(x, y)$:

$$u(x, y) = \int P(x, y)dx + \varphi(y).$$

6 The general solution of the exact differential equation is given by

$$u(x, y) = C.$$

Note:

In Step 3, we can integrate the second equation over the variable y instead of integrating the first equation over x. After integration we need to find the unknown function $\psi(x)$.

Example 1.

Solve the differential equation

$$2xydx + (x^2 + 3y^2)dy = 0.$$

Solution.

The given equation is exact because the partial derivatives are the same:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x.$$

We have the following system of differential equations to find the function $u(x, y)$:

$$\begin{cases} \frac{\partial u}{\partial x} = 2xy \\ \frac{\partial u}{\partial y} = x^2 + 3y^2 \end{cases}.$$

By integrating the first equation with respect to x, we obtain

$$u(x, y) = \int 2xydx = x^2y + \varphi(y).$$

Substituting this expression for $u(x, y)$ into the second equation gives us:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^2y + \varphi(y)\right] = x^2 + 3y^2, \quad \Rightarrow x^2 + \varphi'(y) = x^2 + 3y^2, \quad \Rightarrow \varphi'(y) = 3y^2.$$

By integrating the last equation, we find the unknown function $\varphi(y)$:

$$\varphi(y) = \int 3y^2dy = y^3,$$

so that the general solution of the exact differential equation is given by

$$x^2y + y^3 = C,$$

where C is an arbitrary constant.

Example 2.

Find the solution of the differential equation

$$(6x^2 - y + 3)dx + (3y^2 - x - 2)dy = 0.$$

Solution.

We check this equation for exactness:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3y^2 - x - 2) = -1, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(6x^2 - y + 3) = -1.$$

Hence, the given differential equation is exact. Write the system of equations to determine the function $u(x, y)$:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial x} = P(x, y) = 6x^2 - y + 3 \\ \frac{\partial u}{\partial y} = Q(x, y) = 3y^2 - x - 2 \end{array} \right.$$

Integrate the first equation with respect to the variable x assuming that y is a constant. This produces:

$$u(x, y) = \int (6x^2 - y + 3)dx = \frac{6x^3}{3} - xy + 3x + \varphi(y) = 2x^3 - xy + 3x + \varphi(y).$$

Here we introduced a continuous differentiable function $\varphi(y)$ instead of the constant C.

Plug in the function $u(x, y)$ into the second equation:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[2x^3 - xy + 3x + \varphi(y)\right] = -\cancel{x} + \varphi'(y) = 3y^2 - \cancel{x} - 2.$$

We get equation for the derivative $\varphi'(y)$:

$$\varphi'(y) = 3y^2 - 2.$$

Integrating gives the function $\varphi(y)$:

$$\varphi(y) = \int (3y^2 - 2)dy = y^3 - 2y.$$

So, the function $u(x,y)$ is given by

$$u(x,y) = 2x^3 - xy + 3x + y^3 - 2y.$$

Hence, the general solution of the equation is defined by the following implicit expression:

$$2x^3 - xy + 3x + y^3 - 2y = C,$$

Example 3.

Solve the differential equation

$$e^y dx + (2y + xe^y)dy = 0.$$

Solution.

First we check this equation for exactness:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2y + xe^y) = e^y, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^y) = e^y.$$

We see that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, so that this equation is exact. Find the function $u(x,y)$ from the system of equations:

$$\begin{cases} \frac{\partial u}{\partial x} = e^y \\ \frac{\partial u}{\partial y} = 2y + xe^y \end{cases}$$

Hence,

$$u(x,y) = \int P(x,y)dx = \int e^y dx = xe^y + \varphi(y).$$

Now, by differentiating this expression with respect to y and equating it to $\frac{\partial u}{\partial y}$, we find the derivative $\varphi'(y)$:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[xe^y + \varphi(y)] = 2y + xe^y, \quad \Rightarrow \cancel{xe^y} + \varphi'(y) = 2y + \cancel{xe^y}, \quad \Rightarrow \varphi'(y) = 2y.$$

As a result, we find $\varphi(y)$ and the entire function $u(x,y)$:

$$\varphi(y) = \int 2y\,dy = y^2, \;\; \Rightarrow u(x,y) = xe^y + \varphi(y) = xe^y + y^2.$$

Thus, the general solution of the differential equation is

$$xe^y + y^2 = C.$$

Example 4.

Solve the equation

$$(2xy - \sin x)dx + (x^2 - \cos y)dy = 0.$$

Solution.

This differential equation is exact because

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - \cos y) = 2x = \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy - \sin x) = 2x.$$

We find the function $u(x,y)$ from the system of two equations:

$$\begin{cases} \frac{\partial u}{\partial x} = 2xy - \sin x \\ \frac{\partial u}{\partial y} = x^2 - \cos y \end{cases}.$$

By integrating the 1st equation with respect to the variable x, we have

$$u(x,y) = \int (2xy - \sin x)dx = x^2 y + \cos x + \varphi(y).$$

Plugging in the 2nd equation, we obtain

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^2 y + \cos x + \varphi(y)\right] = x^2 - \cos y,$$

$$\cancel{x^2} + \varphi'(y) = \cancel{x^2} - \cos y, \;\; \Rightarrow \varphi'(y) = -\cos y.$$

Hence,

$$\varphi(y) = \int (-\cos y)dy = -\sin y.$$

Thus, the function $u(x, y)$ is

$$u(x, y) = x^2 y + \cos x - \sin y,$$

so that the general solution of the differential equation is given by the implicit formula:

$$x^2 y + \cos x - \sin y = C.$$

Example 5.

Solve the equation

$$\left(1 + 2x\sqrt{x^2 - y^2}\right)dx - 2y\sqrt{x^2 - y^2}dy = 0.$$

Solution.

First of all we determine whether this equation is exact:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(-2y\sqrt{x^2 - y^2}\right) = -2y \cdot \frac{2x}{2\sqrt{x^2 - y^2}} = -\frac{2xy}{\sqrt{x^2 - y^2}},$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(1 + 2x\sqrt{x^2 - y^2}\right) = 2x \cdot \frac{(-2y)}{2\sqrt{x^2 - y^2}} = -\frac{2xy}{\sqrt{x^2 - y^2}}.$$

As you can see, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. Hence, this equation is exact. Find the function $u(x, y)$, satisfying the system of equations:

$$\begin{cases} \frac{\partial u}{\partial x} = 1 + 2x\sqrt{x^2 - y^2} \\ \frac{\partial u}{\partial y} = -2y\sqrt{x^2 - y^2} \end{cases}.$$

Integrating the first equation gives:

$$u(x,y) = \int \left(1 + 2x\sqrt{x^2 - y^2}\right)dx = x + \frac{\left(x^2 - y^2\right)^{\frac{3}{2}}}{\frac{3}{2}} + \varphi(y) = x + \frac{2}{3}\left(x^2 - y^2\right)^{\frac{3}{2}} + \varphi(y),$$

where $\varphi(y)$ is a certain unknown function of y that will be defined later.

We substitute the result into the second equation of the system:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x + \frac{2}{3}\left(x^2 - y^2\right)^{\frac{3}{2}} + \varphi(y)\right] = -2y\sqrt{x^2 - y^2},$$

$$-\cancel{2y\sqrt{x^2 - y^2}} + \varphi'(y) = -\cancel{2y\sqrt{x^2 - y^2}}, \Rightarrow \varphi'(y) = 0.$$

By integrating the last expression, we find the function $\varphi(y)$:

$$\varphi(y) = C,$$

where C is a constant.

Thus, the general solution of the given differential equation has the form:

$$x + \frac{2}{3}\left(x^2 - y^2\right)^{\frac{3}{2}} + C = 0.$$

Example 6.

Solve the differential equation

$$\frac{1}{y^2} - \frac{2}{x} = \frac{2xy'}{y^3}$$

with the initial condition $y(1) = 1$.

Solution.

Check the equation for exactness by converting it into standard form:

$$\frac{1}{y^2} - \frac{2}{x} = \frac{2x}{y^3}\frac{dy}{dx}, \Rightarrow \left(\frac{1}{y^2} - \frac{2}{x}\right)dx = \frac{2x}{y^3}dy, \Rightarrow \left(\frac{1}{y^2} - \frac{2}{x}\right)dx - \frac{2x}{y^3}dy = 0.$$

The partial derivatives are

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{2x}{y^3}\right) = -\frac{2}{y^3}, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{y^2} - \frac{2}{x}\right) = -\frac{2}{y^3}.$$

Hence, the given equation is exact. Therefore, we can write the following system of equations to determine the function $u(x, y)$:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{1}{y^2} - \frac{2}{x} \\ \frac{\partial u}{\partial y} = -\frac{2x}{y^3} \end{cases}.$$

In the given case, it is more convenient to integrate the second equation with respect to the variable y :

$$u(x, y) = \int \left(-\frac{2x}{y^3}\right) dy = \frac{x}{y^2} + \psi(x).$$

Now we differentiate this expression with respect to the variable x :

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{y^2} + \psi(x)\right] = \frac{1}{y^2} - \frac{2}{x}, \quad \Rightarrow \cancel{\frac{1}{y^2}} + \psi'(x) = \cancel{\frac{1}{y^2}} - \frac{2}{x},$$

$$\psi(x) = -2\ln|x| = \ln\frac{1}{x^2}.$$

Thus, the general solution of the differential equation in implicit form is given by the expression:

$$\frac{x}{y^2} + \ln\frac{1}{x^2} = C.$$

The particular solution can be found using the initial condition $y(1) = 1$. By substituting the initial values, we find the constant C :

$$\frac{1}{1^2} + \ln\frac{1}{1^2} = C, \quad \Rightarrow 1 + 0 = C, \quad \Rightarrow C = 1.$$

Hence, the solution of the given initial value problem is

$$\frac{1}{y^2} + \ln\frac{1}{x^2} = 1.$$