Homogeneous Equation

A first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called homogeneous equation, if the right side satisfies the condition

$$f(tx, ty) = f(x, y)$$

for all t. In other words, the right side is a homogeneous function (with respect to the variables x and y) of the zero order:

Solving Homogeneous Differential Equations

A homogeneous equation can be solved by substitution $y = ux$, which leads to a separable differential equation.

Example 1.

Solve the differential equation

$$(2x + y)dx - xdy = 0.$$

Solution.

It is easy to see that the polynomials $P(x, y)$ and $Q(x, y)$, respectively, at dx and dy, are homogeneous functions of the first order. Therefore, the original differential equation is also homogeneous.

Suppose that $y = ux$, where u is a new function depending on x. Then

$$dy = d(ux) = udx + xdu.$$

Substituting this into the differential equation, we obtain

$$(2x + ux)dx - x(udx + xdu) = 0.$$

Hence,

$$2xdx + \cancel{uxdx} - \cancel{xudx} - x^2du = 0.$$

Dividing both sides by x yields:

$$xdu = 2dx \;\text{ or }\; du = 2\frac{dx}{x}.$$

When dividing by x, we could lose the solution $x = 0$. The direct substitution shows that $x = 0$ is indeed a solution of the given differential equation.

Integrate the latter expression to obtain:

$$\int du = 2\int \frac{dx}{x} \;\text{ or }\; u = 2\ln|x| + C,$$

where C is a constant of integration.

Returning to the old variable y, we can write:

$$y = ux = x\,(2\ln|x| + C).$$

Thus, the equation has two solutions:

$$y = x\,(2\ln|x| + C), \;\; x = 0.$$

Example 2.

Solve the differential equation

$$xy' = y\ln\frac{y}{x}.$$

Solution.

We notice that the root $x = 0$ does not belong to the domain of the differential equation. Rewrite the equation in the form:

$$y' = \frac{y}{x}\ln\frac{y}{x} = f\left(\frac{y}{x}\right).$$

As you can see, this equation is homogeneous.

Make the substitution $y = ux$. Hence,

$$y' = (ux)' = u'x + u.$$

Substituting this expression into the equation gives:

$$x\left(u'x + u\right) = ux \ln \frac{ux}{x}.$$

Divide by $x \neq 0$ to get:

$$u'x + u = u \ln u, \quad \Rightarrow \frac{du}{dx}x = u \ln u - u, \quad \Rightarrow \frac{du}{dx}x = u\left(\ln u - 1\right).$$

We obtain the separable equation:

$$\frac{du}{u\left(\ln u - 1\right)} = \frac{dx}{x}.$$

The next step is to integrate the left and the right side of the equation:

$$\int \frac{du}{u\left(\ln u - 1\right)} = \int \frac{dx}{x},$$

Hence,

$$\ln\left|\ln u - 1\right| = \ln|x| + C.$$

Here the constant C can be written as $\ln C_1$ $(C_1 > 0)$. Then

$$\ln\left|\ln u - 1\right| = \ln|x| + \ln C_1, \quad \Rightarrow \ln\left|\ln u - 1\right| = \ln\left|C_1 x\right|, \quad \Rightarrow \ln u - 1 = \pm C_1 x,$$

$$\ln u = 1 \pm C_1 x \quad \text{or} \quad u = e^{1 \pm C_1 x}.$$

Thus, we have got two solutions:

$$u = e^{1 + C_1 x} \quad \text{and} \quad u = e^{1 - C_1 x}.$$

If $C_1 = 0$, the answer is $y = xe$ and we can make sure that it is also a solution to the equation. Indeed, substituting

$$y = xe, \quad y' = e$$

into the differential equation, we obtain:

$$xe = xe \ln \frac{\cancel{x}\, e}{\cancel{x}}, \quad \Rightarrow xe = xe \ln e, \quad \Rightarrow xe = xe.$$

Then all the solutions can be represented by one formula:

$$y = xe^{1+Cx},$$

where C is an arbitrary real number.

Example 3.

Solve the differential equation

$$(xy + y^2)y' = y^2.$$

Solution.

Here we deal with a homogeneous equation. In fact, we can rewrite it in the form:

$$y' = \frac{y^2}{xy + y^2} = \frac{\frac{y^2}{x^2}}{\frac{xy}{x^2} + \frac{y^2}{x^2}} = \frac{\left(\frac{y}{x}\right)^2}{\frac{y}{x} + \left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right).$$

Make the substitution $y = ux$. Then $y' = u'x + u$. Substituting y and y' into the original equation, we have

$$\left(xux + u^2x^2\right)\left(u'x + u\right) = u^2x^2, \quad \Rightarrow ux^2\left(u + 1\right)\left(u'x + u\right) = u^2x^2.$$

Divide both sides by ux^2. We notice that $x = 0$ is not the solution of the equation. However, one can check that $u = 0$ or $y = 0$ is one of the solutions of the differential equation.

As a result, we have

$$(u + 1)\left(u'x + u\right) = u, \quad \Rightarrow u'x\left(u + 1\right) + u^2 + u = u, \quad \Rightarrow u'x\left(u + 1\right) = -u^2, \quad =$$

$$\left(\frac{1}{u} + \frac{1}{u^2}\right)du = -\frac{dx}{x}.$$

Integrating, we find the general solution:

$$\int \left(\frac{1}{u} + \frac{1}{u^2}\right) du = -\int \frac{dx}{x}, \quad \Rightarrow \ln|u| - \frac{1}{u} = -\ln|x| + C.$$

Taking into account that $u = \frac{y}{x}$, we can write the last expression in the form

$$\ln\left|\frac{y}{x}\right| - \frac{1}{\frac{y}{x}} = -\ln|x| + C, \quad \Rightarrow \ln|y| - \cancel{\ln|x|} - \frac{x}{y} = -\cancel{\ln|x|} + C, \quad \Rightarrow y\ln|y| = Cy + x.$$

The given expression can be represented as an explicit inverse function $x(y)$:

$$x = y\ln|y| - Cy.$$

Since C is an arbitrary real number, we can replace the "minus" sign before the constant to the "plus" sign. Then

$$x = y\ln|y| + Cy.$$

Thus, the given differential equation has the solutions:

$$x = y\ln|y| + Cy, \quad y = 0.$$

Example 4.

Solve the differential equation

$$y' = \frac{y}{x} - \frac{x}{y}.$$

Solution.

As it follows from the right side of the equation, $x \neq 0$ and $y \neq 0$. We can make the substitution $y = ux$, $y' = u'x + u$. This yields:

$$u'x + u = \frac{u\cancel{x}}{\cancel{x}} - \frac{\cancel{x}}{u\cancel{x}}, \quad \Rightarrow u'x + \cancel{u} = \cancel{u} - \frac{1}{u}, \quad \Rightarrow \frac{du}{dx}x = -\frac{1}{u}, \quad \Rightarrow udu = -\frac{dx}{x}.$$

Integrating this equation, we obtain:

$$\int u du = -\int \frac{dx}{x}, \ \Rightarrow \frac{u^2}{2} = -\ln|x| + C, \ \Rightarrow u^2 = 2C - 2\ln|x|.$$

Let the constant $2C$ be denoted by just C. Hence,

$$u^2 = C - 2\ln|x| \ \text{ or } \ u = \pm\sqrt{C - 2\ln|x|}.$$

Thus, the general solution is written in the form

$$y = ux = \pm x\sqrt{C - 2\ln|x|}.$$