Bernoulli Equation

Bernoulli equation is one of the well known nonlinear differential equations of the first order. It is written as

$$y' + a(x)y = b(x)y^m,$$

where $a(x)$ and $b(x)$ are continuous functions.

If $m = 0$, the equation becomes a linear differential equation. In case of $m = 1$, the equation becomes separable.

In general case, when $m \neq 0, 1$, Bernoulli equation can be converted to a linear differential equation using the change of variable

$$z = y^{1-m}.$$

and can be solved by the Linear Differential Equation of First Order.

Example 1.

Find the general solution of the equation

$$y' - y = y^2 e^x.$$

Solution.

We set $m = 2$ for the given Bernoulli equation, so we use the substitution

$$z = y^{1-m} = \frac{1}{y}.$$

Differentiating both sides of the equation (we consider y in the right side as a composite function of x), we obtain:

$$z' = \left(\frac{1}{y}\right)' = -\frac{1}{y^2}y'.$$

Divide both sides of the original differential equation by y^2 :

$$y' - y = y^2 e^x, \quad \Rightarrow \frac{y'}{y^2} - \frac{1}{y} = e^x.$$

Substituting z and z', we find

$$-z - z = e^x, \quad \Rightarrow z' + z = -e^x.$$

We get the linear equation for the function $z(x)$. To solve it, we use the integrating factor:

$$u(x) = e^{\int 1dx} = e^x.$$

Then the general solution of the linear equation is given by

$$z(x) = \frac{\int u(x) f(x) dx + C}{u(x)} = \frac{\int e^x (-e^x) dx + C}{e^x} = \frac{-\frac{e^{2x}}{2} + C}{e^x} = -\frac{e^x}{2} + Ce^{-x} = \frac{2Ce^{-x} - e^x}{2}.$$

Since C is an arbitrary constant, we can replace $2C$ with a constant C_1. Returning to the function $y(x)$, we obtain the implicit expression:

$$y = \frac{1}{z} = \frac{2}{C_1 e^{-x} - e^x}.$$

Note that we have lost the solution $y = 0$ when dividing the equation by y^2. Thus, the final answer is given by

$$y = \frac{2}{C_1 e^{-x} - e^x}, \quad y = 0.$$

Example 2.

Solve the differential equation

$$y' + \frac{y}{x} = y^2.$$

Solution.

As it can be seen, this differential equation is a Bernoulli equation. To solve it, we make the substitution

$$z = y^{1-m} = \frac{1}{y}.$$

Differentiating, we find:

$$z' = \left(\frac{1}{y}\right)' = -\frac{y'}{y^2}.$$

Divide the original equation by y^2 and replace y with z :

$$\frac{y'}{y^2} + \frac{1}{yx} = 1.$$

When dividing by y^2, we have lost the solution $y = 0$. (You can check this by direct substitution.)

In terms of z, the differential equation is written in the form:

$$-z' + \frac{z}{x} = 1 \;\text{ or }\; z' - \frac{z}{x} = -1.$$

We get the linear equation for the function $z(x)$, so we can solve it using the integrating factor technique:

$$u(x) = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\int\frac{dx}{x}} = e^{-\ln|x|} = e^{\ln\frac{1}{|x|}} = \frac{1}{|x|}.$$

We can make sure that the function $\frac{1}{x}$ is the integrating factor. Indeed:

$$z'\cdot\frac{1}{x} - \frac{z}{x}\cdot\frac{1}{x} = z'\cdot\frac{1}{x} - \frac{z}{x^2} = \left(z\cdot\frac{1}{x}\right)'.$$

We see that the left side of the equation becomes the derivative of the product $z(x)u(x)$ after multiplying by $\frac{1}{x}$.

Then the general solution of the linear equation for $z(x)$ is given by

$$z = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int \frac{1}{x} \cdot (-1)dx + C}{\frac{1}{x}} = \frac{-\ln|x| + C}{\frac{1}{x}} = x(C - \ln|x|).$$

Taking into account that $y = \frac{1}{z}$, we can write the answer:

$$y = \frac{1}{x(C - \ln|x|)},$$

or in the implicit form:

$$yx(C - \ln|x|) = 1.$$

Thus, the final answer is

$$yx(C - \ln|x|) = 1, \quad y = 0.$$

Example 3.

Solve the equation

$$y' + y\cot x = y^4 \sin x.$$

Solution.

This is a Bernoulli equation with the parameter $m = 4$. Therefore, we make the substitution $z = y^{1-m} = y^{-3}$. The derivative is given by

$$z' = \left(y^{-3}\right)' = -3y^{-4}y' = -\frac{3y'}{y^4}.$$

Multiply both sides of the original equation by (-3) and divide by y^4 :

$$y' + y\cot x = y^4 \sin x, \quad \Rightarrow -\frac{3y'}{y^4} - \frac{3\cot x}{y^3} = -3\sin x.$$

Notice that in dividing by y^4, we have lost the solution $y = 0$. Rewriting the last equation in terms of z, we get

$$z' - 3\cot x \cdot z = -3\sin x.$$

This differential equation is linear, so we can solve it using the integrating factor:

$$u(x) = e^{\int(-3)\cot x dx} = e^{-3\int \cot x dx} = e^{-3\int \frac{\cos x dx}{\sin x}} = e^{-3\int \frac{d(\sin x)}{\sin x}} = e^{-3\ln|\sin x|} = e^{\ln \frac{1}{|\sin x|^3}} = \frac{1}{|\sin x|^3}.$$

We can take the function $u(x) = \frac{1}{\sin^3 x}$ as the integrating factor. In fact, the left side of the equation becomes the derivative of the product $z(x)u(x)$ after multiplying by $u(x)$:

$$z' \cdot \frac{1}{\sin^3 x} - 3\cot x \cdot z \cdot \frac{1}{\sin^3 z} = z'\frac{1}{\sin^3 z} - \frac{3z\cos x}{\sin^4 x} = \left(z\frac{1}{\sin^3 x}\right)'.$$

Hence, the general solution of the linear differential equation for $z(x)$ can be presented in the form:

$$z = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int \frac{1}{\sin^3 x}(-3\sin x)dx + C}{\frac{1}{\sin^3 x}} = \frac{-3\int \frac{dx}{\sin^2 x} + C}{\frac{1}{\sin^3 x}} = (3\cot x + C)\sin^3 x.$$

Since $z = y^{-3}$, we obtain the following solutions of the given Bernoulli equation:

$$\frac{1}{y^3} = (3\cot x + C)\sin^3 x, \quad y = 0.$$