## Second Order Differential Equations

A second order differential equation is written in general form as

$$F\left(x, y, y', y''\right) = 0,$$

where $F$ is a function of the given arguments.

If the differential equation can be resolved for the second derivative $y''$, it can be represented in the following explicit form:

$$y'' = f\left(x, y, y'\right).$$

## Second Order Linear Homogeneous Differential Equations with Constant Coefficients

Consider a differential equation of type

$$y'' + py' + qy = 0,$$

where $p, q$ are some constant coefficients.

For each of the equation we can write the so-called characteristic (auxiliary) equation:

$$k^2 + pk + q = 0.$$

The general solution of the homogeneous differential equation depends on the roots of the characteristic quadratic equation. There are the following options:

1 $k_1$ and $k_2$ are real numbers and $k_1 \neq k_2$

$$y\left(x\right) = C_1 e^{k_1 x} + C_2 e^{k_2 x},$$

where $C_1$ and $C_2$ are arbitrary real numbers.

2 $k_1$ and $k_2$ are real numbers and $k_1 = k_2$

$$y(x) = (C_1 x + C_2) e^{k_1 x}.$$

3 $k_1$ and $k_2$ are complex numbers

$$y(x) = e^{\alpha x} [C_1 \cos(\beta x) + C_2 \sin(\beta x)].$$

## Solving Quadratic Equations by Quadratic Formula

A quadratic equation is an equation that could be written as:

$$ax^2 + bx + c = 0$$

The method of solving quadratic equations involves the use of the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Example 1.**

Solve the differential equation

$$y'' - 6y' + 5y = 0.$$

*Solution.*

First we write the corresponding characteristic equation for the given differential equation:

$$k^2 - 6k + 5 = 0.$$

Eliminate the constant $C$ from the system of equations:

$$y(x) = C_1 e^x + C_2 e^{5x},$$

where $C_1$ and $C_2$ are arbitrary constants.

**Example 2.**

Find the general solution of the equation

$$y'' - 6y' + 9y = 0.$$

*Solution.*

We write the characteristic equation and calculate its roots:

$$k^2 - 6k + 9 = 0, \qquad k_{1,2} = 3.$$

As it can be seen, the characteristic equation has one root of order $2 : k_1 = 3$. Therefore, the general solution of the differential equation is given by

$$y(x) = (C_1 x + C_2) e^{3x},$$

where $C_1$, $C_2$ are arbitrary real numbers.

**Example 3.**

Solve the differential equation

$$y'' - 4y' + 5y = 0.$$

*Solution.*

First we write the corresponding characteristic equation and find its roots:

$$k^2 - 4k + 5 = 0, \qquad k_{1,2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

Thus, the differential equation has a pair of complex conjugate roots: $k_1 = 2 + i$, $k_2 = 2 - i$. In this case, the general solution is expressed by the formula:

$$y(x) = e^{2x} \left[ C_1 \cos x + C_2 \sin x \right],$$

where $C_1$, $C_2$ are arbitrary constants.