Second Order Linear Nonhomogeneous Differential Equations with Constant Coefficients

The nonhomogeneous differential equation of this type has the form

$$y'' + py' + qy = f(x),$$

where p, q are constant numbers (that can be both as real as complex numbers). For each equation we can write the related homogeneous or complementary equation:

$$y'' + py' + qy = 0.$$

The soulution will be:

$$y = C_1(x)Y_1(x) + C_2(x)Y_2(x)$$

satisfies the nonhomogeneous equation with the right side $f(x)$.

$Y_1(x)$, $Y_2(x)$: the functions times C_1 and C_2 in the homogeneous solution

The unknown functions $C_1(x)$ and $C_2(x)$ can be determined from the system of two equations:

$$\begin{cases} C_1'(x)Y_1(x) + C_2'(x)Y_2(x) = 0 \\ C_1'(x)Y_1'(x) + C_2'(x)Y_2'(x) = f(x) \end{cases}.$$

Example 1.

Solve the differential equation

$$y'' + y = \sin(2x).$$

Solution.

First we solve the related homogeneous equation $y'' + y = 0$. The roots of the corresponding characteristic equation are purely imaginary:

$$k^2 + 1 = 0, \quad \Rightarrow k^2 = -1, \quad \Rightarrow k_{1,2} = \pm i.$$

$$y(x) = C_1(x)\cos x + C_2(x)\sin x,$$

The functions $C_1(x)$ and $C_2(x)$ can be determined from the following system of equations:

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0 \\ C_1'(x)(\cos x)' + C_2'(x)(\sin x)' = \sin 2x \end{cases}$$

Then

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0 \\ C_1'(x)(-\sin x) + C_2'(x)\cos x = \sin 2x \end{cases}$$

We can express the derivative $C_1'(x)$ from the first equation:

$$C_1'(x) = -C_2'(x)\frac{\sin x}{\cos x}.$$

Substituting this in the second equation, we find the derivative $C_2'(x)$:

$$\left(-C_2'(x)\frac{\sin x}{\cos x}\right)(-\sin x) + C_2'(x)\cos x = \sin 2x, \quad \Rightarrow C_2'(x)\left(\frac{\sin^2 x}{\cos x} + \cos x\right) = \sin 2x$$

$$C_2'(x)\frac{\sin^2 x + \cos^2 x}{\cos x} = \sin 2x, \quad \Rightarrow C_2'(x)\frac{1}{\cos x} = \sin 2x, \quad \Rightarrow C_2'(x) = \sin 2x \cos x.$$

$$C_1'(x) = -\sin 2x \cos x \cdot \frac{\sin x}{\cos x} = -\sin 2x \sin x.$$

Integrating the expressions for the derivatives $C_1'(x)$ and $C_2'(x)$, gives

$$C_1(x) = \int(-\sin 2x \sin x)dx = -2\int \sin^2 x \cos x dx = -2\cdot\frac{\sin^3 x}{3} + A_1 = -\frac{2}{3}\sin^3 x + A_1,$$

$$C_2(x) = \int(\sin 2x \cos x)dx = 2\int \sin x \cos^2 x dx = -2\cdot\frac{\cos^3 x}{3} + A_2 = -\frac{2}{3}\cos^3 x + A_2.$$

$$y(x) = C_1(x)\cos x + C_2(x)\sin x = \left(-\frac{2}{3}\sin^3 x + A_1\right)\cos x + \left(-\frac{2}{3}\cos^3 x + A_2\right)\sin x$$

$$= A_1\cos x + A_2\sin x - \frac{2}{3}\sin^3 x\cos x - \frac{2}{3}\cos^3 x \sin x$$

$$= A_1\cos x + A_2\sin x - \frac{2}{3}\sin x\cos x\left(\underbrace{\sin^2 x + \cos^2 x}_{1}\right)$$

$$= A_1\cos x + A_2\sin x - \frac{1}{3}\cdot 2\sin x\cos x = A_1\cos x + A_2\sin x - \frac{1}{3}\sin 2x.$$