Second Order Euler Equation

A second order linear differential equation of the form

$$x^2y'' + Axy' + By = 0, \quad x > 0$$

is called the Euler differential equation. It can be reduced to the linear homogeneous differential equation with constant coefficients.

We make the following substitution: $x = e^t$. Then the derivatives will be

$$y' = e^{-t}\frac{dy}{dt},$$

$$y'' = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Putting this into the original Euler equation gives:

$$\frac{d^2y}{dt^2} + (A-1)\frac{dy}{dt} + By = 0.$$

As it can be seen, we obtain the linear equation with constant coefficients. The corresponding characteristic equation has the form:

$$k^2 + (A-1)k + B = 0.$$

Now we can determine the roots of the characteristic equation and write the general solution for the function $y(t)$. Then we can easily return to the function $y(x)$ taking into account that

$$y(t) = y(\ln x).$$

Example 1.

Find the general solution of the differential equation

$$4x^2y'' + y = 0$$

assuming that $x > 0$.

Solution.

We make the substitution $x = e^t$. As

$$y'' = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right),$$

the equation becomes:

$$4\,\cancel{e^{2t}}\,\cancel{e^{-2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + y = 0, \quad \Rightarrow 4\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + y = 0.$$

Calculate the roots of the corresponding characteristic equation:

$$4k^2 - 4k + 1 = 0, \quad \Rightarrow \quad k = \frac{1}{2}.$$

The equation has one root of order 2. Then the general solution for the function $y(t)$ is given by

$$y(t) = (C_1 + C_2t)e^{\frac{t}{2}}.$$

Now it's easy to write the solution for the original function $y(x)$:

$$y(x) = (C_1 + C_2\ln x)e^{\frac{\ln x}{2}} = (C_1 + C_2\ln x)x^{\frac{1}{2}} = (C_1 + C_2\ln x)\sqrt{x},$$

where C_1 and C_2 are arbitrary real numbers.

Example 2.

Find the general solution of the Euler equation

$$x^2y'' + xy' + y = 5x^2$$

for $x > 0$.

Solution.

First we construct the general solution of the homogeneous equation:

$$x^2y'' + xy' + y = 0.$$

Make the substitution:

$$x = e^t, \quad y' = e^{-t}\frac{dy}{dt}, \quad y'' = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

As a result the homogeneous differential equation is written in the form:

$$\cancel{e^{2t}}\,\cancel{e^{-2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + \cancel{e^{t}}\,\cancel{e^{-t}}\frac{dy}{dt} + y = 0, \quad \Rightarrow \frac{d^2y}{dt^2} - \cancel{\frac{dy}{dt}} + \cancel{\frac{dy}{dt}} + y = 0,$$

$$\frac{d^2y}{dt^2} + y = 0.$$

Solve the characteristic equation:

$$k^2 + 1 = 0, \quad \Rightarrow k_{1,2} = \pm i.$$

Thus, the roots of the characteristic equation are imaginary. Therefore, the general solution of the homogeneous equation is given by

$$y_0(t) = C_1 \cos t + C_2 \sin t,$$

where C_1 and C_2 are real constants.

Determine a particular solution of the non-homogeneous equation:

$$\frac{d^2y}{dt^2} + y = 5e^{2t}.$$

Taking into account the structure of the right side, we look for a particular solution of the form $y_1(t) = ae^{2t}$ where a is a constant coefficient. Then

$$\frac{dy_1}{dt} = 2ae^{2t}, \quad \frac{d^2y_1}{dt^2} = 4ae^{2t}.$$

Substitute the function and its derivative into the equation to find the coefficient a :

$$4ae^{2t} + ae^{2t} = 5e^{2t}, \quad \Rightarrow 5ae^{2t} = 5e^{2t}, \quad \Rightarrow a = 1.$$

Thus, a particular solution of the non-homogeneous equation is given by

$$y_1(t) = e^{2t}.$$

Now we can write the general solution of the non-homogeneous equation:

$$y(t) = y_0(t) + y_1(t) = C_1 \cos t + C_2 \sin t + e^{2t}.$$

Going back to the variable x, we obtain

$$y(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x) + e^{2\ln x}.$$

As $e^{2\ln x} = e^{\ln x^2} = x^2$, the final answer is written in the form

$$y(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x) + x^2.$$