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Multiple Integrals and Applications

Double Integrals in Cartesian Coordinates

The definite integral can be extended to functions of more than one variable. Consider, for example, a function of two variables $z = f(x, y)$. The double integral of function $f(x, y)$ is denoted by

$$\iint\limits_R f(x, y)dA,$$

where R is the region of integration in the xy-plane.

If the definite integral $\int_a^b f(x)dx$ of a function of one variable $f(x) \geq 0$ is the area under the curve $f(x)$ from $x = a$ to $x = b$, then the double integral is equal to the volume under the surface $z = f(x, y)$ and above the xy-plane in the region of integration R (Figure 1).

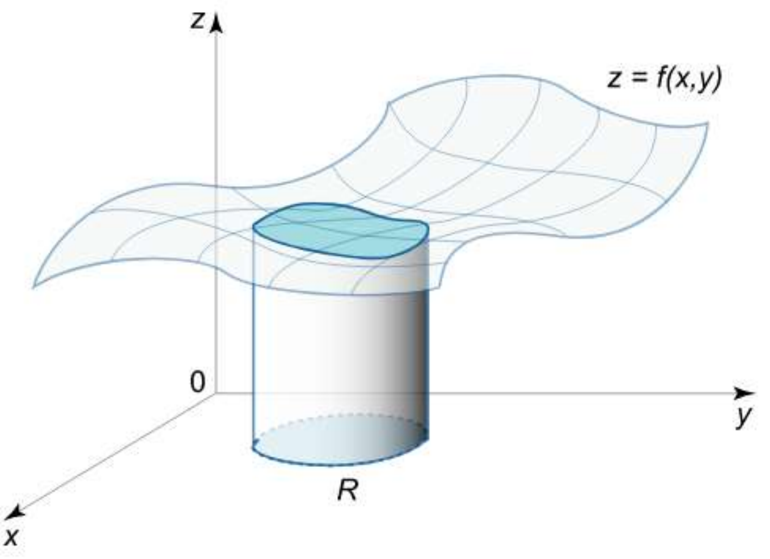

Figure 1.

Regions of Type I and Type II

The most powerful tool for the evaluation of the double integrals is the Fubini's theorem. It works not for a general region R but for some special regions which we call Regions of type I or type II.

Definition 1.

A plane region R is said to be of type I if it lies between the graphs of two continuous functions of x (Figure 1), that is

$$R = \{(x, y) |\ a \le x \le b,\ \ p(x) \le y \le q(x)\}.$$

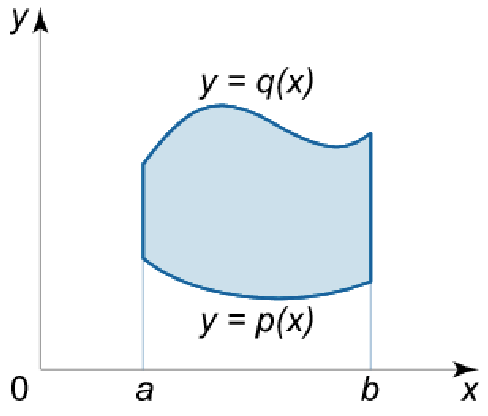

Figure 1.

Definition 2.

A plane region R is said to be of type II if it lies between the graphs of two continuous functions of y (Figure 2), that is

$$R = \{(x, y) |\ u(y) \le x \le v(y),\ \ c \le y \le d\}.$$

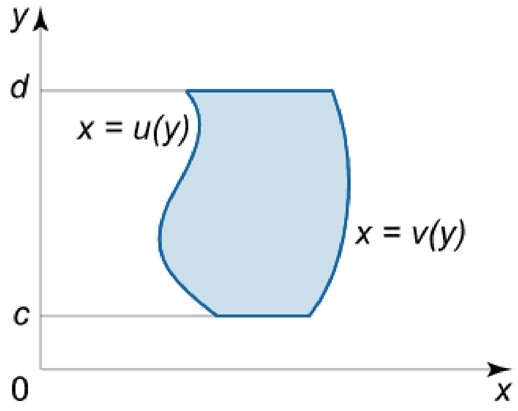

Figure 2.

Fubini's Theorem

Let $f(x, y)$ is a continuous function over a type I region R such that

$$R = \{(x, y) |\ a \le x \le b,\ \ p(x) \le y \le q(x)\}.$$

Then the double integral of $f(x, y)$ in this region is expressed in terms of the iterated integral:

$$\iint\limits_R f(x, y) dA = \int\limits_a^b \int\limits_{p(x)}^{q(x)} f(x, y) dy dx.$$

For a region of type II we have the similar theorem:

If $f(x, y)$ is a continuous function on a type II region R such that

$$R = \{(x, y) |\ u(y) \le x \le v(y),\ \ c \le y \le d\}.$$

then

$$\iint\limits_R f(x, y) dA = \int\limits_c^d \int\limits_{u(y)}^{v(y)} f(x, y) dx dy.$$

Thus, the Fubini's theorem allows to calculate double integrals through the iterated ones. To evaluate an iterated integral, we first find the inner integral and then the outer integral.

Example 1.

Evaluate the iterated integral

$$\int_0^1 \int_1^2 xydydx.$$

Solution.

We first evaluate the inner integral and then the outer integral:

$$\int_0^1 \int_1^2 xydydx = \int_0^1 \left[\int_1^2 xydy \right] dx = \int_0^1 \left[x \left(\frac{y^2}{2} \right) \Big|_1^2 \right] dx = \int_0^1 x \frac{3}{2} dx = \frac{3}{2} \left(\frac{x^2}{2} \right) \Big|_0^1 = \frac{3}{4}.$$

Example 2.

Find the iterated integral

$$\int_0^1 \int_y^{y^2} (x+2y)dxdy.$$

Solution.

Here we have the region of type II. Applying the Fubini's theorem we obtain

$$\int_0^1 \int_y^{y^2} (x+2y)dxdy = \int_0^1 \left[\int_y^{y^2} (x+2y)dx \right] dy = \int_0^1 \left[\left(\frac{x^2}{2} + 2yx \right) \Big|_y^{y^2} \right] dy$$

$$= \int_0^1 \left[\left(\frac{y^4}{2} + 2y^3 \right) - \left(\frac{y^2}{2} + 2y^2 \right) \right] dy = \int_0^1 \left[\frac{y^4}{2} + 2y^3 - \frac{5y^2}{2} \right] dy = \left[\frac{y^5}{10} + \frac{y^4}{2} - \frac{5y^3}{6} \right] \Big|_0^1$$

$$= \frac{1}{10} + \frac{1}{2} - \frac{5}{6} = -\frac{7}{30}.$$