Double Integrals in Polar Coordinates

One of the particular cases of change of variables is the transformation from Cartesian to polar coordinate system (Figure 1) :

$$x = r\cos\theta, \quad y = r\sin\theta.$$

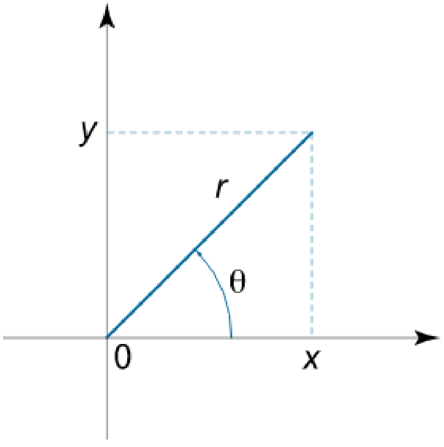


Figure 1.

Let the region R in polar coordinates be defined as follows (Figure 2):

$$0 \le g(\theta) \le r \le h(\theta), \quad \alpha \le \theta \le \beta, \quad \text{where} \quad \beta - \alpha \le 2\pi.$$

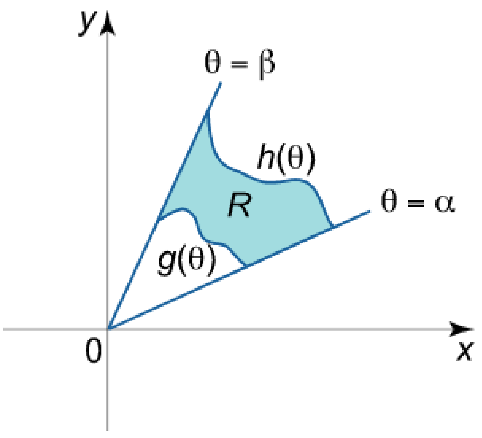


Figure 2.

Then the double integral in polar coordinates is given by the formula

$$\iint_R f(x,y)dxdy = \int_{\alpha}^{\beta}\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

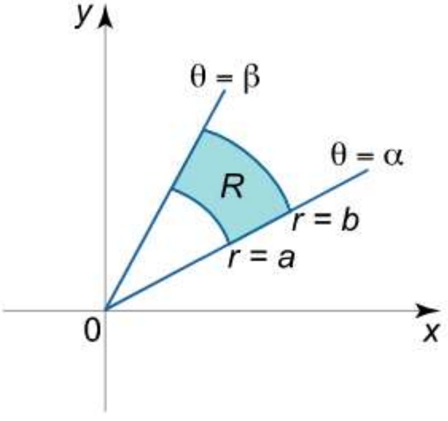


Figure 3.

The region of integration (Figure 3) is called the polar rectangle if it satisfies the following conditions:

$$0 \le a \le r \le b, \quad \alpha \le \theta \le \beta, \quad \text{where} \quad \beta - \alpha \le 2\pi.$$

In this case the formula for change of variables can be written as

$$\iint_R f(x,y)dxdy = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Example 1.

Calculate the double integral

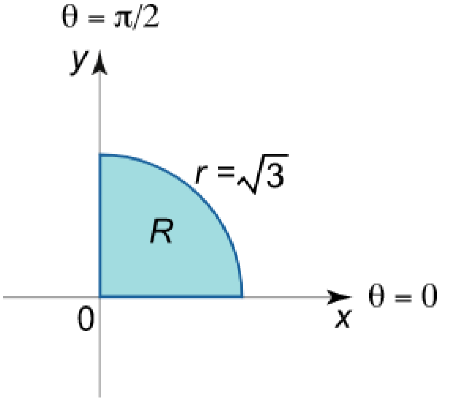
$$\iint\limits_R (x^2+y^2)\,dydx$$

by transforming to polar coordinates. The region of integration R is the sector $0 \le \theta \le \frac{\pi}{2}$ of a circle with radius $r = \sqrt{3}$.

Solution.

The region R is the polar rectangle

$$R = \left\{ (r,\theta) \mid 0 \le r \le \sqrt{3},\ 0 \le \theta \le \frac{\pi}{2} \right\}.$$



Using the formula

$$\iint\limits_R f(x,y)\,dxdy = \int\limits_\alpha^\beta \int\limits_a^b f(r\cos\theta, r\sin\theta)\,r\,drd\theta,$$

we obtain

$$\iint\limits_R (x^2+y^2)\,dydx = \int\limits_0^{\frac{\pi}{2}} \int\limits_0^{\sqrt{3}} r^2\left(\cos^2\theta + \sin^2\theta\right) r\,drd\theta = \int\limits_0^{\frac{\pi}{2}} d\theta \int\limits_0^{\sqrt{3}} r^3 dr = \theta\Big|_0^{\frac{\pi}{2}} \cdot \left(\frac{r^4}{4}\right)\Bigg|_0^{\sqrt{3}}$$

$$= \frac{\pi}{2} \cdot \frac{9}{4} = \frac{9\pi}{8}.$$

Example 2.

Evaluate the integral

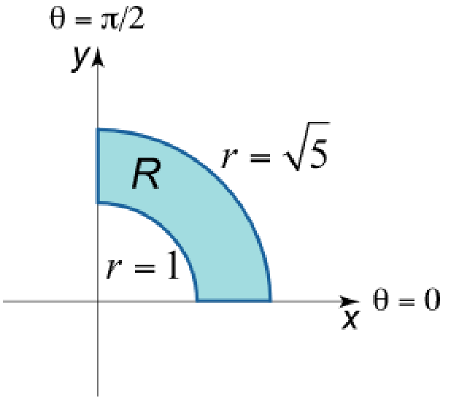
$$\iint_R xydydx,$$

where the region of integration R lies in the sector $0 \le \theta \le \frac{\pi}{2}$ between the curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$.

Solution.

In polar coordinates, the region of integration R is the polar rectangle

$$R = \left\{ (r, \theta) \mid 1 \le r \le \sqrt{5},\ 0 \le \theta \le \frac{\pi}{2} \right\}.$$



So using the formula

$$\iint_R f(x, y) dxdy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta,$$

we find the integral:

$$\iint_R xydydx = \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{5}} r\cos\theta r\sin\theta r dr d\theta = \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta \int_1^{\sqrt{5}} r^3 dr = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta \int_1^{\sqrt{5}} r^3 dr$$

$$= \frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{2}} \cdot \left(\frac{r^4}{4}\right)\Bigg|_1^{\sqrt{5}} = \frac{1}{4}(-\cos\pi + \cos 0) \cdot \frac{1}{4}(25 - 1) = \frac{1}{4}(1 + 1) \cdot 6 = 3.$$