Area by Double Integrals in Cartesian Coordinates

If $f(x,y) = 1$ in the integral $\iint\limits_R f(x,y)dxdy$, then the double integral gives the area of the region R.

The area of a type I region (Figure 1) can be written in the form:

$$A = \int_a^b \int_{g(x)}^{h(x)} dydx.$$

Similarly, the area of a type II region (Figure 2) is given by the formula

$$A = \int_c^d \int_{p(y)}^{q(y)} dxdy.$$

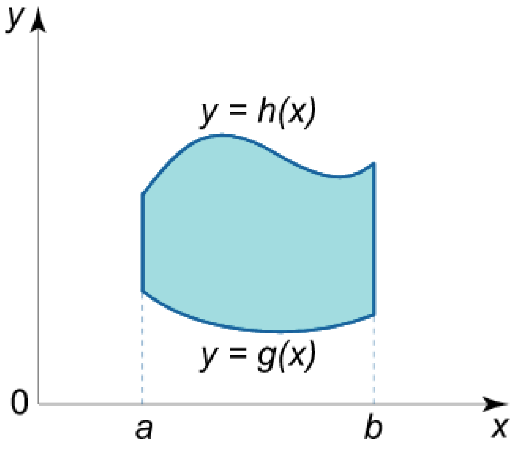

Figure 1.

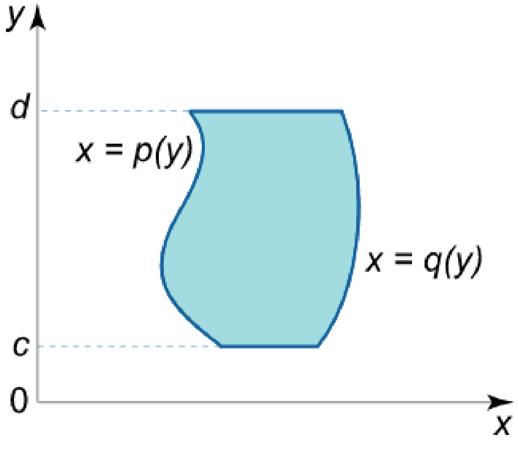

Figure 2.

Volume by Double Integrals in Cartesian Coordinates

If $f(x,y) > 0$ over a region R, then the volume of the solid below the surface $z = f(x,y)$ and above R is expressed as

$$V = \iint_R f(x,y)dA.$$

If R is a type I region bounded by $x = a$, $x = b$, $y = g(x)$, $y = h(x)$, the volume of the solid is

$$V = \iint_R f(x,y)dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx.$$

Similarly, if R is a type II region bounded by $y = c$, $y = d$, $x = p(y)$, $x = q(y)$, the volume of the solid is given by

$$V = \iint_R f(x,y)dA = \int_c^d \int_{p(y)}^{q(y)} f(x,y)dxdy.$$

If $f(x,y) \geq g(x,y)$ over a region R, then the volume of the cylindrical solid between the surfaces $z_1 = g(x,y)$ and $z_2 = f(x,y)$ over R is given by

$$V = \iint_R [f(x,y) - g(x,y)]dA.$$

Surface Area by Double Integrals in Cartesian Coordinates

We assume that the surface is given as a graph of function $z = g(x,y)$, and the domain of this function is a region R. Then the area of the surface over the region R is

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy,$$

provided that the derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are continuous over the region R.

Areas and Volumes by Double Integrals in Polar Coordinates

If S is a region in the xy-plane bounded by $\theta=\alpha, \theta=\beta, r=h(\theta), r=g(\theta)$ (Figure 3), then the area of the region is defined by the formula

$$A=\iint_R dA=\int_{\alpha}^{\beta}\int_{h(\theta)}^{g(\theta)} r dr d\theta.$$

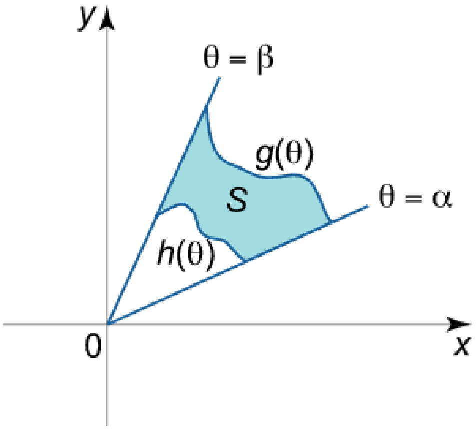

Figure 3.

The volume of the solid below $z=f(r,\theta)$ over a region S in polar coordinates is given by

$$V=\iint_S f(r,\theta) r dr d\theta.$$

Example 1.

Find the area of the region R bounded by the hyperbolas

$$y = \frac{a^2}{x}, y = \frac{2a^2}{x}\ (a > 0)$$

and the vertical lines $x = 1, x = 2$.

Solution.

The region R is sketched in Figure 4.

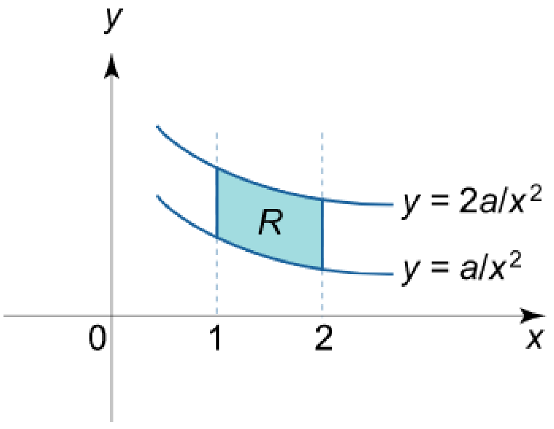

Figure 4.

Using the formula for the area of a type I region

$$A = \iint_R dxdy = \int_a^b \int_{g(x)}^{h(x)} dydx,$$

we have

$$A = \iint_R dxdy = \int_1^2 \left[\int_{\frac{a^2}{x}}^{\frac{2a^2}{x}} dy \right] dx = \int_1^2 \left[y|_{\frac{a^2}{x}}^{\frac{2a^2}{x}} \right] dx = \int_1^2 \left(\frac{2a^2}{x} - \frac{a^2}{x} \right) dx$$

$$= a^2 \int_1^2 \frac{dx}{x} = a^2 (\ln 2 - \ln 1) = a^2 \ln 2.$$

Example 2.

Find the area of the region R bounded by

$$y^2 = a^2 - ax, y = a + x.$$

Solution.

We first determine the points of intersection of the two curves.

$$\begin{cases} y^2 = a^2 - ax \\ y = a + x \end{cases}, \ \Rightarrow (a+x)^2 = a^2 - ax, \ \Rightarrow a^2 + 2ax + x^2 = a^2 - ax, \ \Rightarrow x^2 + 3ax = 0,$$

$$\Rightarrow x(x + 3a) = 0, \ \Rightarrow x_{1,2} = 0; \ -3a.$$

So the coordinates of the points of intersection are

$$x_1 = 0, \ \ y_1 = a + 0 = a,$$

$$x_2 = -3a, \ \ y_2 = a - 3a = -2a.$$

It is simpler to consider R as a type II region (Figure 5).

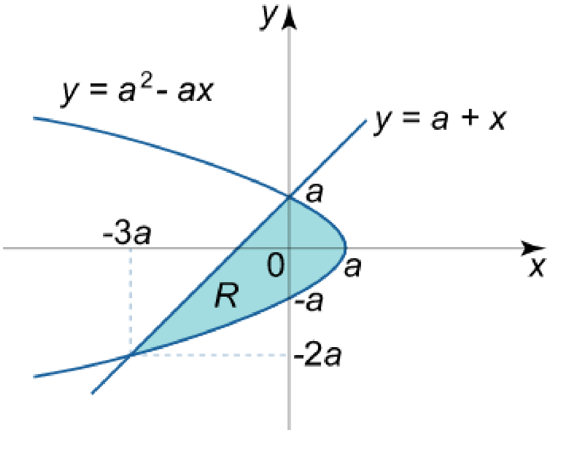

Figure 5.

To calculate the area of the region, we transform the equations of the boundaries:

$$y^2 = a^2 - ax, \ \Rightarrow ax = a^2 - y^2, \ \Rightarrow x = a - \frac{y^2}{a},$$

$$y = a + x, \ \Rightarrow x = y - a.$$

Then we have

$$A = \iint\limits_R dxdy = \int\limits_{-2a}^{a}\left[\int\limits_{y-a}^{a-\frac{y^2}{a}} dx\right] dy = \int\limits_{-2a}^{a}\left[\int\limits_{y-a}^{a-\frac{y^2}{a}} dx\right] dy = \int\limits_{-2a}^{a}\left[x\Big|_{y-a}^{a-\frac{y^2}{a}}\right] dy$$

$$= \int\limits_{-2a}^{a}\left[a - \frac{y^2}{a} - (y-a)\right] dy = \int\limits_{-2a}^{a}\left(2a - \frac{y^2}{a} - y\right) dy = \left(2ay - \frac{y^3}{3a} - \frac{y^2}{2}\right)\Big|_{-2a}^{a}$$

$$= \left(2a^2 - \frac{a^3}{3a} - \frac{a^2}{2}\right) - \left(-4a^2 + \frac{8a^3}{3a} - \frac{4a^2}{2}\right) = \frac{9a^2}{2}.$$

Example 3.

Find the volume of the solid in the first octant bounded by the planes

$$y = 0, z = 0, z = x, z + x = 4.$$

Solution.

The given solid is shown in Figure 6.

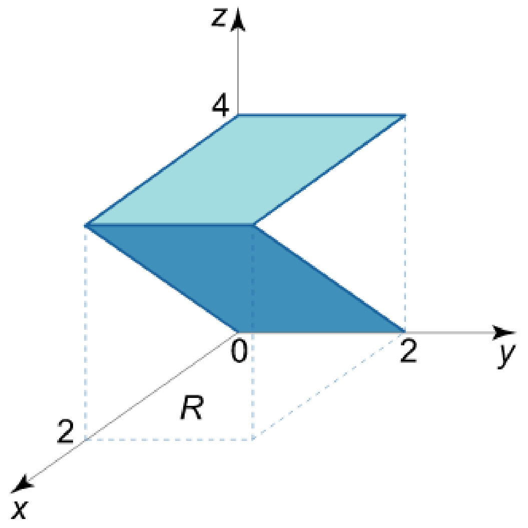

Figure 6.

As it can be seen from the figure, the base R is the square in the first quadrant. For given x and y, the z-value in the solid varies from $z = x$ to $z = 4 - x$. Then the volume is

$$V = \iint_R [(4-x)-x]dxdy = \int_0^2 \left[\int_0^2 (4-2x)dy\right] dx = \int_0^2 \left[(4y-2xy)\big|_{y=0}^{2}\right] dx = \int_0^2 (8-4x)dx$$

$$= \left(8x - 2x^2\right)\Big|_0^2 = 16 - 8 = 8.$$

Example 4.

Describe the solid whose volume is given by the integral

$$V = \int_0^1 dx \int_0^{1-x} \left(x^2 + y^2\right) dy.$$

Solution.

The given solid (Figures $7, 8$) lies above the triangle R in the xy-plane, bounded by the coordinate axes Ox, Oy and the straight line $y = 1 - x$, and under the paraboloid $z = x^2 + y^2$.

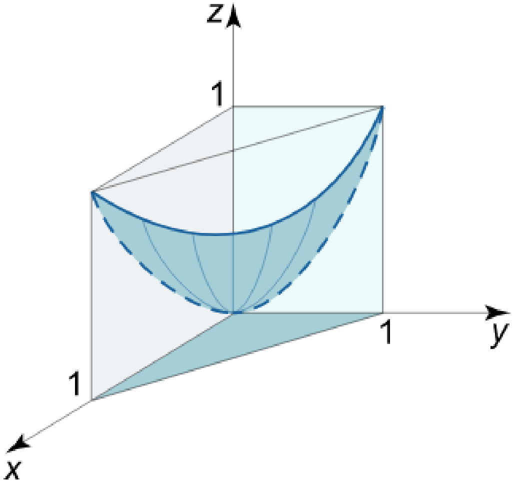

Figure 7.

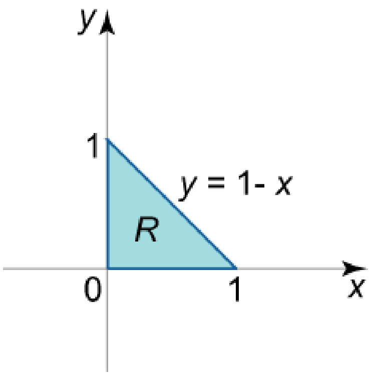

Figure 8.

The volume of the solid is

$$V = \int_0^1 dx \int_0^{1-x} (x^2 + y^2) dy = \int_0^1 \left[\left(x^2 y + \frac{y^3}{3} \right) \Big|_{y=0}^{1-x} \right] dx = \int_0^1 \left[x^2 (1-x) + \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \left(2x^2 - \frac{4x^3}{3} - x + \frac{1}{3} \right) dx = \left(\frac{2x^3}{3} - \frac{4}{3} \cdot \frac{x^4}{4} - \frac{x^2}{2} + \frac{x}{3} \right) \Big|_0^1 = \frac{2}{3} - \frac{1}{3} - \frac{1}{2} + \frac{1}{3} = \frac{1}{6}.$$