Example 5.

Find the volume of the solid bounded by

$$z = xy, x + y = a, z = 0.$$

Solution.

The solid lies above the triangle R in the xy-plane (see Figures 9, 10) and under the surface $z = xy$.

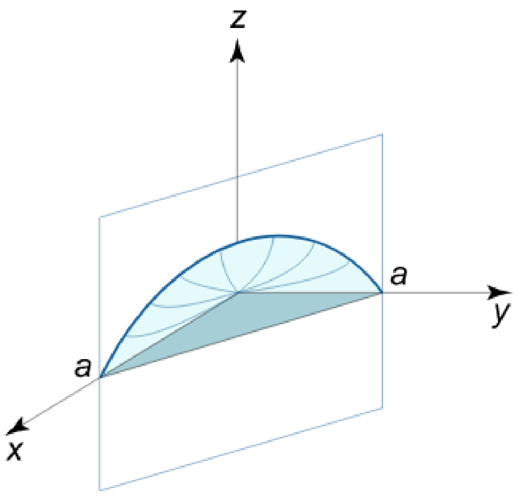

Figure 9.

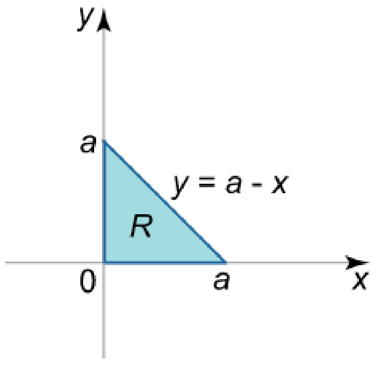

Figure 10.

The volume of the solid is

$$V = \iint\limits_R xydxdy = \int\limits_0^a \left[\int\limits_0^{a-x} xydy \right] dx = \int\limits_0^a \left[\left(\frac{xy^2}{2} \right) \Big|_{y=0}^{a-x} \right] dx = \frac{1}{2} \int\limits_0^a x(a-x)^2 dx$$

$$= \frac{1}{2}\int_0^a x\left(a^2 - 2ax + x^2\right)dx = \frac{1}{2}\int_0^a \left(a^2x - 2ax^2 + x^3\right)dx = \frac{1}{2}\left(a^2 \cdot \frac{x^2}{2} - 2a \cdot \frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_0^a$$

$$= \frac{1}{2}\left(\frac{a^2}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{a^4}{24}.$$

Example 6.

Calculate the surface area of a sphere of radius a.

Solution.

Consider the upper hemisphere of the sphere. Its equation is

$$x^2 + y^2 + z^2 = a^2 \text{ or } z = \sqrt{a^2 - x^2 - y^2}.$$

Obviously, the region of integration R is the disk of the same radius a centered at the origin. The surface area of the hemisphere is given by formula

$$S_{\frac{1}{2}} = \iint\limits_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$$

Calculate the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\sqrt{a^2 - x^2 - y^2} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2 - x^2 - y^2}} = -\frac{x}{z},$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\sqrt{a^2 - x^2 - y^2} = \frac{-\cancel{2}y}{\cancel{2}\sqrt{a^2 - x^2 - y^2}} = -\frac{y}{z}.$$

Substituting these derivatives, we have

$$S_{\frac{1}{2}} = \iint\limits_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dxdy = \iint\limits_R \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\,dxdy$$

$$= \iint\limits_R \sqrt{\frac{z^2 + x^2 + y^2}{z^2}}\,dxdy = \iint\limits_R \frac{a}{z}\,dxdy.$$

Convert the double integral into polar coordinates.

$$S_{\frac{1}{2}} = \iint\limits_R \frac{a}{z} dxdy = \int\limits_0^{2\pi} \int\limits_0^a \frac{a}{\sqrt{a^2 - r^2}} r dr d\theta = a \int\limits_0^{2\pi} d\theta \int\limits_0^a \frac{r dr}{\sqrt{a^2 - r^2}}$$

$$= -2\pi a \int\limits_0^a \frac{d\left(a^2 - r^2\right)}{2\sqrt{a^2 - r^2}} = -2\pi a \left(\sqrt{a^2 - r^2} \right) \Big|_{r=0}^{a} = -2\pi a \left(0 - a\right) = 2\pi a^2.$$

So the surface area of the entire sphere is

$$S = 2S_{\frac{1}{2}} = 4\pi a^2.$$

Example 7.

Find the area of one loop of the rose defined by the equation

$$r = \cos 2\theta.$$

Solution.

We consider the loop in the sector $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (Figure 13).

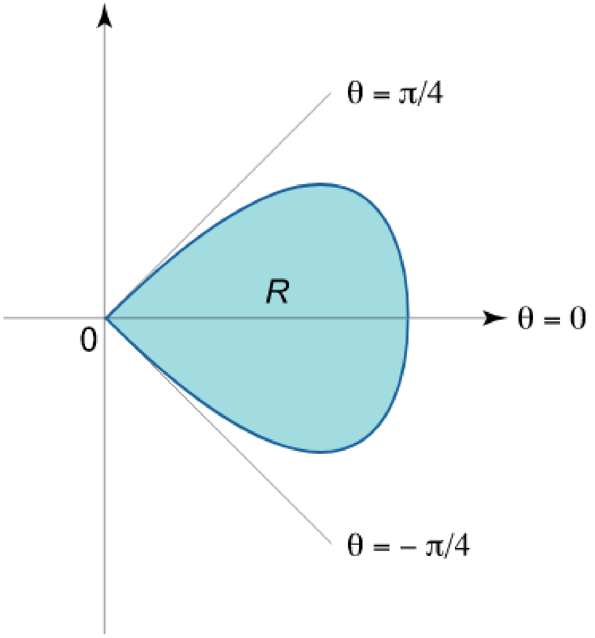

Figure 13.

The region of integration R can be written in the form

$$R = \left\{(r,\theta) \mid 0 \le r \le \cos 2\theta,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}.$$

Hence, the area of the region in polar coordinates is

$$A = \iint_R r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^{\cos 2\theta} r\,dr\right] d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\left(\frac{r^2}{2}\right)\Big|_0^{\cos 2\theta}\right] d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{2}\,d\theta = \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 4\theta)\,d\theta = \frac{1}{4}\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4} + \frac{\sin \pi}{4}\right) - \left(-\frac{\pi}{4} + \frac{\sin(-\pi)}{4}\right)\right] = \frac{\pi}{8}.$$