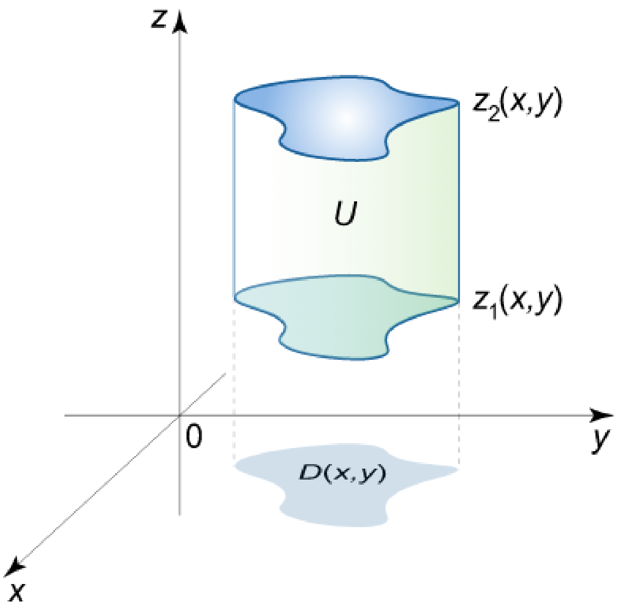
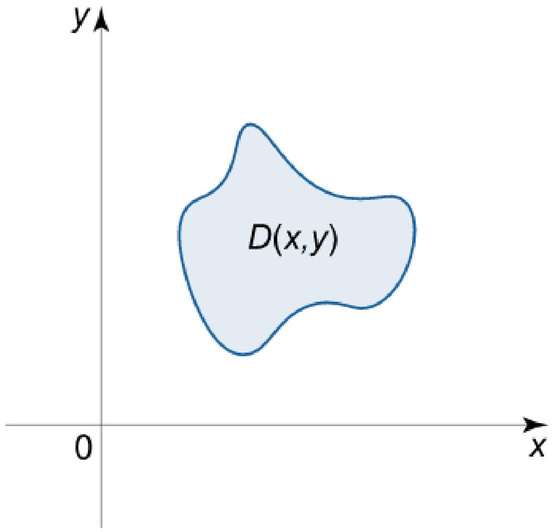
## Triple Integrals in Cartesian Coordinates

Calculation of a triple integral in Cartesian coordinates can be reduced to the consequent calculation of three integrals of one variable.

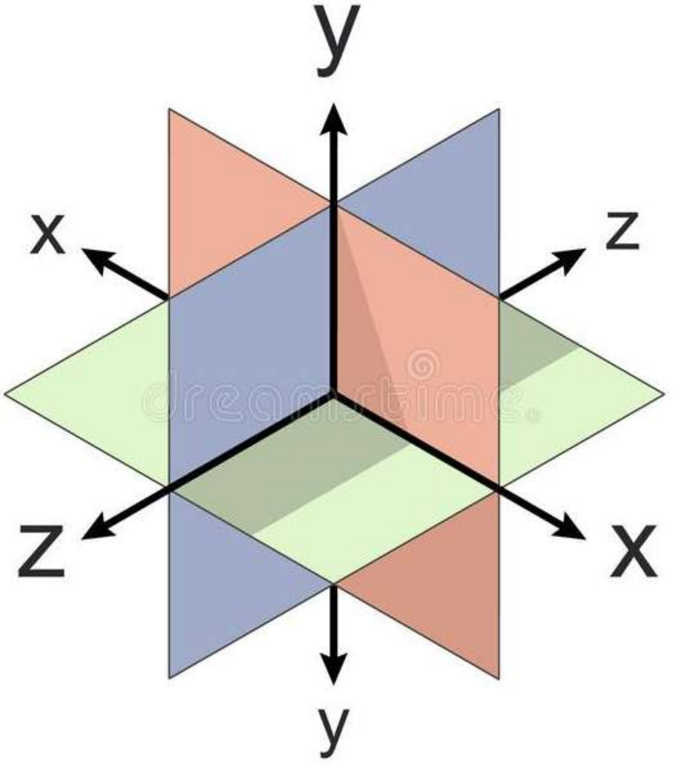
Consider the case when a three dimensional region $U$ is a type I region, i.e. any straight line parallel to the $z$-axis intersects the boundary of the region $U$ in no more than 2 points. Let the region $U$ be bounded below by the surface $z = z_1(x, y)$, and above by the surface $z = z_2(x, y)$ .




The projection of the solid $U$ onto the $xy$-plane is the region $D$ .



| Octants | X | Y | Z |
|---|---|---|---|
| First | + | + | + |
| Second | - | + | + |
| Third | - | - | + |
| Fourth | + | - | + |
| Fifth | + | - | - |
| Sixth | - | - | - |
| Seventh | - | + | - |
| Eighth | + | + | - |



## Fubini's Theorem to Triple Integral

In the particular case when the region of integration $U$ is the rectangular box $[a,b]\times[c,d]\times[p,q]$, the triple integral is given by

$$\iiint\limits_U f(x,y,z)\,dxdydz = \int_a^b dx \int_c^d dy \int_p^q f(x,y,z)\,dz.$$

**Example 1.**

Evaluate the integral

$$\int_0^2 \int_0^z \int_0^y xyz\,dxdydz.$$

*Solution.*

Applying the Fubini's theorem, we can calculate the iterated integral starting from the inner one:

$$I = \int_0^2 \int_0^z \int_0^y xyz\,dxdydz = \int_0^2 dz \int_0^z dy \int_0^y xyz\,dx = \int_0^2 dz \int_0^z dy \left[\left(\frac{x^2yz}{2}\right)\Bigg|_{x=0}^{x=y}\right] = \int_0^2 dz \int_0^z \frac{y^3z}{2}\,dy =$$

$$= \frac{1}{2}\int_0^2 dz \int_0^z y^3z\,dy = \frac{1}{2}\int_0^2 dz \left[\left(\frac{y^4z}{4}\right)\Bigg|_{y=0}^{y=z}\right] = \frac{1}{2}\int_0^2 \frac{z^5}{4}\,dz = \frac{1}{8}\int_0^2 z^5\,dz = \frac{1}{8}\left(\frac{z^6}{6}\right)\Bigg|_0^2 = \frac{64}{48}$$

$$= \frac{4}{3}.$$

**Example 2.**

Evaluate the integral

$$\iiint\limits_U (1-x)\,dx\,dy\,dz,$$

where the region $U$ lies in the first octant below the plane

$$3x+2y+z=6.$$

*Solution.*

The limits of integrations for $z$ range from $z=0$ to $z=6-3x-2y$. Considering the projection $D$ in the $xy$-plane, we find that the variable $y$ ranges from $y=0$ to $y=3-\frac{3}{2}x$ , while the variable $x$ runs from 0 to 2.

Consequently, the triple integral is expressed through iterated integral as

$$I=\iiint\limits_U (1-x)\,dx\,dy\,dz=\int\limits_0^2 dx\int\limits_0^{3-\frac{3}{2}x} dy\int\limits_0^{6-3x-2y}(1-x)\,dz.$$

Calculate successively the three integrals to get

$$I=\int\limits_0^2 dx\int\limits_0^{3-\frac{3}{2}x} dy\int\limits_0^{6-3x-2y}(1-x)\,dz=\int\limits_0^2 dx\int\limits_0^{3-\frac{3}{2}x} dy\left[(z-zx)\big|_{z=0}^{z=6-3x-2y}\right]$$

$$=\int\limits_0^2 dx\int\limits_0^{3-\frac{3}{2}x}[6-3x-2y-(6-3x-2y)x]\,dy$$

$$=\int\limits_0^2 dx\int\limits_0^{3-\frac{3}{2}x}(6-3x-2y-6x+3x^2+2xy)\,dy=\int\limits_0^2 dx\int\limits_0^{3-\frac{3}{2}x}(6-9x-2y+3x^2+2xy)\,dy$$

$$=\int\limits_0^2\left(9-18x+\frac{45}{4}x^2-\frac{9}{4}x^3\right)dx=\left(9x-\frac{18}{2}x^2+\frac{45}{12}x^3-\frac{9}{16}x^4\right)\Big|_0^2=18-36+30-9=3$$

**Example 3.**

Calculate the triple integral

$$\iiint\limits_U xy^2z^3\,dxdydz,$$

where the region $U$ is bounded by the surfaces

$$z = xy, y = x, x = 0, x = 1, z = 0.$$

*Solution.*

$$I = \iiint\limits_U xy^2z^3\,dxdydz = \int\limits_0^1 dx \int\limits_0^x dy \int\limits_0^{xy} xy^2z^3\,dz = \int\limits_0^1 dx \int\limits_0^x dy \left[ \left( \frac{xy^2z^4}{4} \right) \Big|_{z=0}^{z=xy} \right]$$

$$= \int\limits_0^1 dx \int\limits_0^x \left( xy^2 \frac{x^4y^4}{4} \right) dy = \frac{1}{4} \int\limits_0^1 dx \int\limits_0^x x^5y^6\,dy = \frac{1}{4} \int\limits_0^1 dx \left[ \left( \frac{x^5y^7}{7} \right) \Big|_{y=0}^{y=x} \right] = \frac{1}{4} \int\limits_0^1 \left( x^5 \frac{x^7}{7} \right) dx$$

$$= \frac{1}{28} \int\limits_0^1 x^{12}\,dx = \frac{1}{28} \left( \frac{x^{13}}{13} \right) \Big|_0^1 = \frac{1}{28} \cdot \frac{1}{13} = \frac{1}{364}.$$

## Triple Integrals in Cylindrical Coordinates

The position of a point $M\,(x, y, z)$ in the $xyz$-space in cylindrical coordinates is defined by three numbers: $\rho, \varphi, z$, where $\rho$ is the projection of the radius vector of the point $M$ onto the $xy$-plane, $\varphi$ is the angle formed by the projection of the radius vector with the $x$-axis (Figure 1), $z$ is the projection of the radius vector on the $z$-axis (its value is the same in Cartesian and cylindrical coordinates).

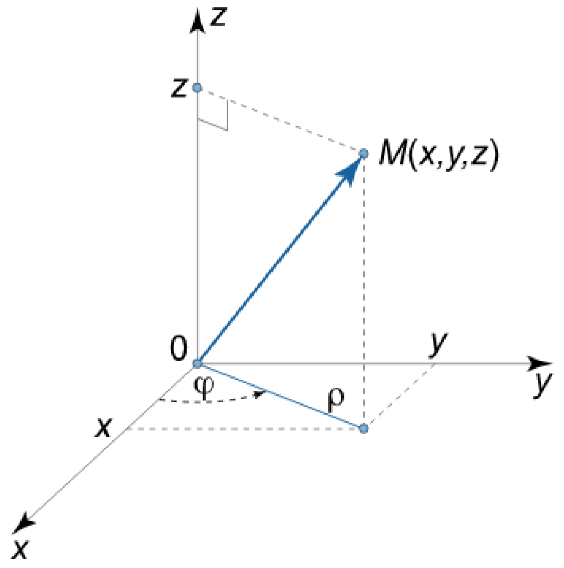



Figure 1.

The relationship between cylindrical and Cartesian coordinates of a point is given by

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z.$$

We assume here that

$$\rho \geq 0, \quad 0 \leq \varphi \leq 2\pi, \quad -\infty < z < \infty.$$

Then the formula of change of variables for this transformation can be written in the form

$$\iiint\limits_{U} f\,(x, y, z) dx dy dz = \iiint\limits_{U'} f\,(\rho \cos \varphi, \rho \sin \varphi, z) \rho d\rho d\varphi dz.$$

Transition from cylindrical coordinates makes calculation of triple integrals simpler in those cases when the region of integration is formed by a cylindrical surface.

**Example 1.**

Evaluate the integral

$$\iiint\limits_U (x^4 + 2x^2y^2 + y^4)\,dxdydz,$$

where the region $U$ is bounded by the surface $x^2 + y^2 \leq 1$ and the planes $z = 0$, $z = 1$ (Figure 2).

*Solution.*

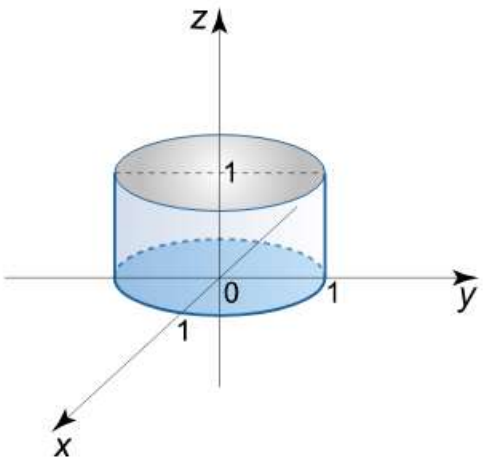



Figure 2.

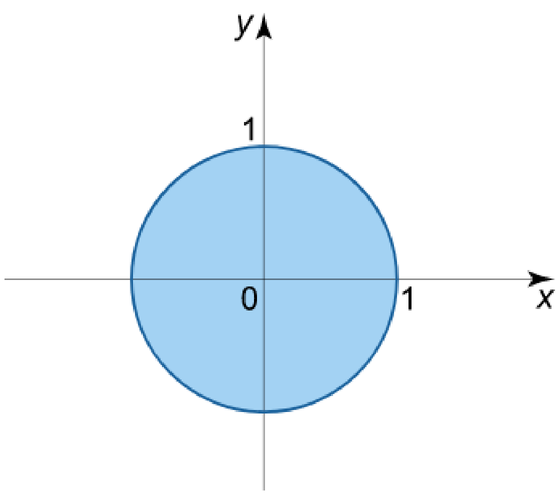



Figure 3.

It is more convenient to calculate this integral in cylindrical coordinates. Projection of the region of integration onto the $xy$-plane is the circle $x^2 + y^2 \leq 1$ or $0 \leq \rho \leq 1$ (Figure 3).

Notice that the integrand can be written as

$$\left(x^4+2x^2y^2+y^4\right)=\left(x^2+y^2\right)^2=\left(\rho^2\right)^2=\rho^4.$$

Then the integral becomes

$$I=\int_0^{2\pi}d\varphi\int_0^1\rho^4\rho d\rho\int_0^1 dz.$$

$$I=\int_0^{2\pi}d\varphi\int_0^1\rho^4\rho d\rho\int_0^1 dz=2\pi\int_0^1\rho^5 d\rho\int_0^1 dz=2\pi\cdot 1\cdot\int_0^1\rho^5 d\rho=2\pi\left(\frac{\rho^6}{6}\right)\Big|_0^1=2\pi\cdot\frac{1}{6}=\frac{\pi}{3}.$$

## Triple Integrals in Spherical Coordinates

The spherical coordinates of a point $M\,(x, y, z)$ are defined to be the three numbers: $\rho, \varphi, \theta$, where

- $\rho$ is the length of the radius vector to the point $M$;
- $\varphi$ is the angle between the projection of the radius vector $\overrightarrow{OM}$ on the $xy$-plane and the $x$-axis;
- $\theta$ is the angle of deviation of the radius vector $\overrightarrow{OM}$ from the positive direction of the $z$-axis (Figure 1).

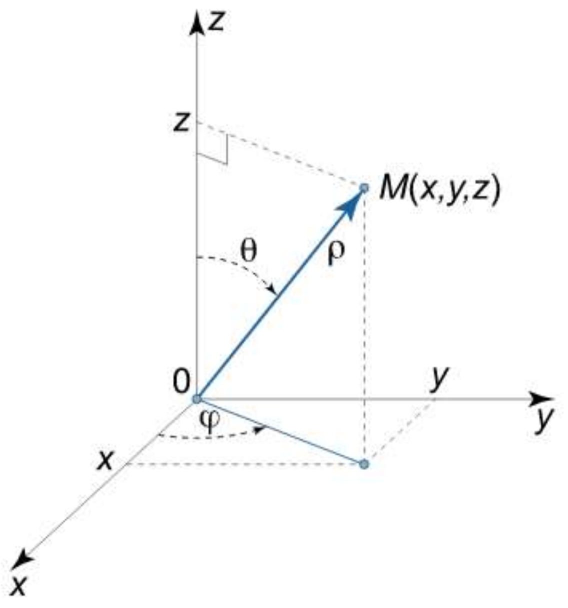



Figure 1.

The spherical coordinates of a point are related to its Cartesian coordinates as follows:

$$x = \rho \cos \varphi \sin \theta, \quad y = \rho \sin \varphi \sin \theta, \quad z = \rho \cos \theta,$$

$$\text{where} \quad \rho \geq 0, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta \leq \pi.$$

Hence, the formula of change of variables for this transformation is

$$\iiint\limits_U f\,(x, y, z) dx dy dz = \iiint\limits_{U'} f\,(\rho \cos \varphi \sin \theta,\ \rho \sin \varphi \sin \theta,\ \rho \cos \theta) \rho^2 \sin \theta d\rho d\varphi d\theta.$$

It is easier to calculate triple integrals in spherical coordinates when the region of integration $U$ is a ball (or some portion of it) and/or when the integrand is a kind of $f\left(x^2 + y^2 + z^2\right)$.

### Example 1.

Evaluate the integral

$$\iiint\limits_{U} \sqrt{x^2 + y^2 + z^2}dxdydz,$$

where the region of integration $U$ is the ball given by the equation

$$x^2 + y^2 + z^2 = 25.$$

*Solution.*

As the region $U$ is a ball and the integrand is expressed by a function depending on $f\left(x^2 + y^2 + z^2\right)$, we can convert the triple integral to spherical coordinates. Make the substitution:

$$x = \rho\cos\varphi\sin\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\theta,$$

The new variables range within the limits:

$$0 \le \rho \le 5, \quad 0 \le \varphi \le 2\pi, \quad 0 \le \theta \le \pi.$$

$$I = \iiint\limits_{U} \sqrt{x^2 + y^2 + z^2}dxdydz = \iiint\limits_{U'} \rho \cdot \rho^2 \sin\theta d\rho d\varphi d\theta = \int\limits_0^{2\pi} d\varphi \int\limits_0^5 \rho^3 d\rho \int\limits_0^{\pi} \sin\theta d\theta = \int\limits_0^{2\pi} d\varphi \int\limits_0^5$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_0^5 \rho^3 d\rho \left[\left(-\cos\theta\right)\Big|_0^{\pi}\right] = \int\limits_0^{2\pi} d\varphi \int\limits_0^5 \rho^3 d\rho \left(-\cos\pi + \cos 0\right) = 2\int\limits_0^{2\pi} d\varphi \int\limits_0^5 \rho^3 d\rho$$

$$= 2\int\limits_0^{2\pi} d\varphi \cdot \left[\left(\frac{\rho^4}{4}\right)\Bigg|_0^5\right] = 2\int\limits_0^{2\pi} d\varphi \cdot \frac{5^4}{4} = \frac{625}{2}\int\limits_0^{2\pi} d\varphi = \frac{625}{2} \cdot 2\pi = 625\pi.$$