Volumes by Triple Integrals

The volume of a solid U in Cartesian coordinates xyz is given by

$$V = \iiint\limits_U dx dy dz.$$

In cylindrical coordinates, the volume of a solid is defined by the formula

$$V = \iiint\limits_U \rho d\rho d\varphi dz.$$

In spherical coordinates, the volume of a solid is expressed as

$$V = \iiint\limits_U \rho^2 \sin\theta d\rho d\varphi d\theta.$$

Example 1.

Find the volume of a cone of height H and base radius R (Figure 1).

Solution.

The cone is bounded by the surface $z = \frac{H}{R}\sqrt{x^2+y^2}$ and the plane $z = H$ (see Figure 1).

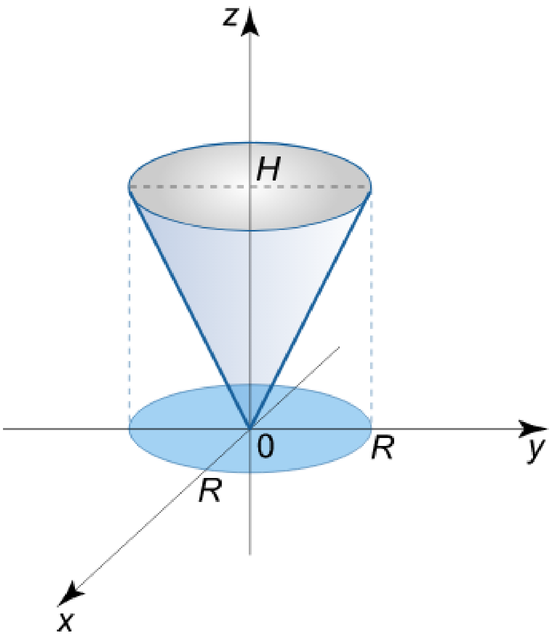

Figure 1.

Its volume in Cartesian coordinates is expressed by the formula

$$V = \iiint\limits_U dxdydz = \int\limits_{-R}^{R} dx \int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int\limits_{\frac{H}{R}\sqrt{x^2+y^2}}^{H} dz.$$

Calculate this integral in cylindrical coordinates that range within the limits:

$$0 \le \varphi \le 2\pi, \;\; 0 \le \rho \le R, \;\; \rho \le z \le H.$$

As a result, we obtain (do not forget to include the Jacobian ρ):

$$V = \int\limits_0^R \rho d\rho \int\limits_0^{2\pi} d\varphi \int\limits_{\frac{H}{R}\rho}^{H} dz.$$

Then the volume of the cone is

$$V = \int_0^R \rho d\rho \int_0^{2\pi} d\varphi \int_{\frac{H}{R}\rho}^{H} dz = 2\pi \int_0^R \rho d\rho \int_{\frac{H}{R}\rho}^{H} dz = 2\pi \int_0^R \rho d\rho \cdot \left[z|_{z=\frac{H}{R}\rho}^{z=H} \right] = 2\pi \int_0^R \rho \left(H - \frac{H}{R}\rho \right) d\rho$$

$$= 2\pi \int_0^R \left(H\rho - \frac{H}{R}\rho^2 \right) d\rho = 2\pi \left[\left(\frac{\rho^2 H}{2} - \frac{\rho^3 H}{3R} \right) \Bigg|_{\rho=0}^{\rho=R} \right] = 2\pi \left(\frac{R^2 H}{2} - \frac{R^3 H}{3R} \right) = \frac{2\pi R^2 H}{6}$$

$$= \frac{\pi R^2 H}{3}.$$

Example 2.

Find the volume of the ball

$$x^2 + y^2 + z^2 \le R^2.$$

Solution.

We calculate the volume of the part of the ball lying in the first octant $(x \ge 0, y \ge 0, z \ge 0)$, and then multiply the result by 8. This yields:

$$V = 8 \iiint_U dx dy dz = 8 \iiint_{U'} \rho^2 \sin\theta d\rho d\varphi d\theta = 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^R \rho^2 d\rho \int_0^{\frac{\pi}{2}} \sin\theta d\theta$$

$$= 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^R \rho^2 d\rho \cdot \left[(-\cos\theta)|_0^{\frac{\pi}{2}} \right] = 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^R \rho^2 d\rho \cdot \left(-\cos\frac{\pi}{2} + \cos 0 \right) = 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^R \rho^2 d\rho \cdot 1$$

$$= 8 \int_0^{\frac{\pi}{2}} d\varphi \cdot \left[\left(\frac{\rho^3}{3} \right) \Bigg|_0^R \right] = \frac{8R^3}{3} \int_0^{\frac{\pi}{2}} d\varphi = \frac{8R^3}{3} \cdot \left[\varphi|_0^{\frac{\pi}{2}} \right] = \frac{8R^3}{3} \cdot \frac{\pi}{2} = \frac{4\pi R^3}{3}.$$

Example 3.

Find the volume of the tetrahedron bounded by the planes passing through the points $A\,(1,0,0)$, $B\,(0,2,0)$, $C\,(0,0,3)$ and the coordinate planes Oxy, Oxz, Oyz (Figure 2).

Solution.

The equation of the straight line AB in the xy-plane (Figure 3) is written as $y = 2 - 2x$. The variable x ranges here in the interval $0 \le x \le 1$, and the variable y ranges in the interval $0 \le y \le 2 - 2x$.

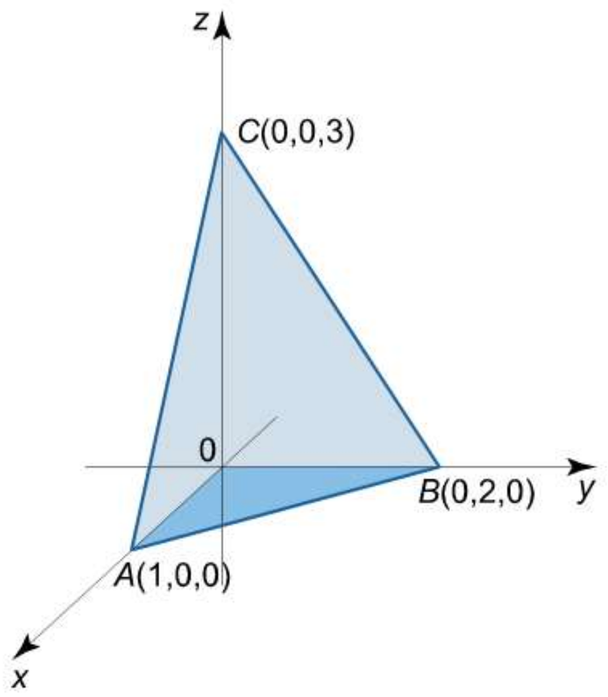

Figure 2.

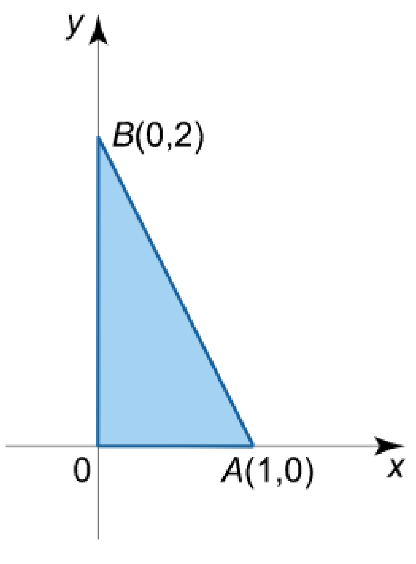

Figure 3.

We write the equation of the plane ABC in segment form. Since the plane ABC cuts the line segments $1, 2,$ and 3, respectively, on the $x-$, $y-$, and $z-$axis, then its equation can be written as

$$\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1.$$

In general case the equation of the plane ABC is written as

$$6x+3y+2z=6 \text{ or } z=3-3x-\frac{3}{2}y.$$

Hence, the limits of integration over the variable z range in the interval from $z=0$ to $z=3-3x-\frac{3}{2}y$.

Now we can calculate the volume of the tetrahedron:

$$V=\iiint_U dxdydz=\int_0^1 dx\int_0^{2-2x} dy\int_0^{3-3x-\frac{3}{2}y} dz=\int_0^1 dx\int_0^{2-2x} dy\cdot\left[z\Big|_0^{3-3x-\frac{3}{2}y}\right]$$

$$=\int_0^1 dx\int_0^{2-2x}\left(3-3x-\frac{3}{2}y\right)dy=\int_0^1 dx\cdot\left[\left(3y-3xy-\frac{3}{4}y^2\right)\Big|_{y=0}^{y=2-2x}\right]$$

$$=\int_0^1\left[3(2-2x)-3x(2-2x)-\frac{3}{4}(2-2x)^2\right]dx$$

$$=\int_0^1\left[6-6x-6x+6x^2-\frac{3}{4}(4-8x+4x^2)\right]dx=\int_0^1(6-12x+6x^2-3+6x-3x^2)dx$$

$$=3\int_0^1(1-2x+x^2)dx=3\left[\left(x-x^2+\frac{x^3}{3}\right)\Big|_0^1\right]=3\cdot\left(\cancel{1}-\cancel{1^2}+\frac{1^3}{3}\right)=1.$$