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Matrices
And
Determinants

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Matrices and Systems of Equations

Matrix is a rectangular array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

$$a_{row,column}$$

Each entry is an element.

Augmented Matrix is two matrices combined together.

Order of matrix (Dimension) is represent:

$$\text{Rows} \times \text{columns}$$

What is the order of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$?

The answer is:

2×3

Reduced Row Echelon Form

We have seen that every linear system of equations can be written in matrix form. For example, the system

$$x + 2y + 3z = 4$$
$$3x + 4y + z = 5$$
$$2x + y + 3z = 6$$

can be written as

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

is called the **matrix of coefficients** of the system. It is also useful to form the **augmented matrix**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 5 \\ 2 & 1 & 3 & 6 \end{bmatrix}.$$

Note that the fourth column consists of the numbers in the system on the right side of the equal signs.

Definition 1. *A matrix is in* **row echelon form** *if*

1. Nonzero rows appear above the zero rows.

2. In any nonzero row, the first nonzero entry is a one (called the **leading one***).*

3. The leading one in a nonzero row appears to the left of the leading one in any lower row.

Definition 2. *A matrix is in* **reduced row echelon form (RREF)** *if the three conditions in Definition 1 hold and in addition, we have*

4. If a column contains a leading one, then all the other entries in that column are zero.

Example.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form, but not in RREF.

Example.

$$\begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF.

Example.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF.

Example.

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF.

Any matrix can be transformed into its RREF by performing a series of operations on the rows of the matrix. The general plan is to first transform the entries in the lower left into zeros. The final step is to transform all the entries above the leading ones into zeros. The allowable operations are called **elementary row operations**. They are:

1. Divide a row by a nonzero number.
2. Subtract a multiple of a row from another row.
3. Interchange two rows.

Performing any of these operations on an augmented matrix leads to a new system of equations which has the same set of solutions as the original system.

Example. Consider the system

$$2x + 8y + 4z = 2$$
$$2x + 5y + z = 5$$
$$4x + 10y - z = 1.$$

The corresponding augmented matrix is

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

Step 1. $\frac{1}{2}$ (Row 1) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

Step 2. (Row 2) $-$ 2 (Row 1) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

Step 3. (Row 3) $-$ 4 (Row 1) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{bmatrix}$$

Stpe 4. $-\frac{1}{3}$ (Row 2) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -6 & -9 & -3 \end{bmatrix}$$

Step 5. (Row 3) + 6 (Row 2) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

Step 6. $-\frac{1}{3}$ (Row 3) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 7. (Row 1) − 4 (Row 2) gives

$$\begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 8. (Row 2) − (Row 3) gives

$$\begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 9. (Row 1) + 2 (Row 3) gives

$$\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This matrix is in RREF.

Once the augmented matrix of a linear system is put into RREF, is is easy to find all the solutions. A column of the matrix which contains a leading one is called a **leading column**. A variable which corresponds to a leading column is called a **leading variable**. The non-leading variables are called **free variables**. To find all solutions to the system, just solve the equations for the leading variables in terms of the free variables.

Example. Continuing with the previous example, suppose that the RREF is

$$\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

This has 3 leading variables and no free variables. The linear system corresponding to the RREF is $x_1 = 11$, $x_2 = -4$, $x_3 = 3$. The equations are already solved for the leading variables. The system has the one solution $(11, -4, 3)$.

Example. Suppose that the RREF of the augmented matrix of a linear system is

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding system is

$$x_1 + x_3 + x_4 = 3$$
$$x_2 + 2x_4 = 1.$$

The leading variables are x_1, x_2. The free variables are x_3, x_4. Solving for the leading variables gives

$$x_1 = -x_3 - x_4 + 3$$
$$x_2 = -2x_4 + 1.$$

Once the free variables are assigned values, we obtain a solution (x_1, x_2, x_3, x_4). The solution can also be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 + 3 \\ -2x_4 + 1 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$