Matrix Operations

Matrix Addition and Subtraction

- Both matrices must have same order
- Add or subtract corresponding elements

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -4 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -2 & -3 \\ -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 3+0 & 1+(-1) \\ 0+(-2) & 2+(-3) \\ -4+(-4) & -1+(-5) \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ -2 & -1 \\ -8 & -6 \end{bmatrix}$$

Scalar Multiplication

- Multiply a matrix with a number
- Distribute

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 3\cdot 1 & 3\cdot 2 & 3\cdot 3 \\ 3\cdot 0 & 3\cdot -1 & 3\cdot -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 9 \\ 0 & -3 & -6 \end{bmatrix}$$

Matrix Multiplication

- Number of columns in 1st = number of rows in 2nd
- $(m \times n) \cdot (n \times p)$
- Order of product m × p
- Order is important

$$\begin{bmatrix} 2 & -1 & 7 \\ 0 & 6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \cdot 0 + (-1)(-2) + 7 \cdot 3 \\ 0 \cdot 0 + 6 \cdot (-2) + (-3) \cdot 3 \end{bmatrix}$$

$$\begin{bmatrix} 23 \\ -21 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2(-1) + 0(-2) & 2(0) + 0(1) & 2(4) + 0(2) \\ 1(-1) + 3(-2) & 1(0) + 3(1) & 1(4) + 3(2) \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 8 \\ -7 & 3 & 10 \end{bmatrix}$$

Inverse Matrices

- Identity Matrix (I)
- $A \cdot I = A$
- $A \cdot A^{-1} = I$
- Both A and A^{-1} must be square

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

OR

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

OR

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of 2×2

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Find the inverse of $\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$

$$\frac{1}{1(4)-(-2)(0)}\begin{bmatrix} 4 & -0 \\ 2 & 1 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1/2 & 1/4 \end{bmatrix}$$

Augment the matrix with the Identity Matrix

Turn the original matrix into the identity matrix

$$[A \vdots I] \rightarrow [I \vdots A^{-1}]$$

Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -3 & 4 & -4 \end{bmatrix}$

Augment with identity

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ -3 & 4 & -4 & 0 & 0 & 1 \end{array}\right]$$

3×1st add to 3rd

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 10 & 5 & 3 & 0 & 1 \end{array}\right]$$

10×2nd add to 3rd

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -15 & 3 & 10 & 1 \end{array}\right]$$

-2×3rd add to 15×2nd

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -15 & 0 & -6 & -5 & -2 \\ 0 & 0 & -15 & 3 & 10 & 1 \end{array}\right]$$

3rd add to 5×1st

$$\left[\begin{array}{ccc:ccc} 5 & 10 & 0 & 8 & 10 & 1 \\ 0 & -15 & 0 & -6 & -5 & -2 \\ 0 & 0 & -15 & 3 & 10 & 1 \end{array}\right]$$

2×2nd add to 3×1st

$$\left[\begin{array}{ccc:ccc} 15 & 0 & 0 & 12 & 20 & -1 \\ 0 & -15 & 0 & -6 & -5 & -2 \\ 0 & 0 & -15 & 3 & 10 & 1 \end{array}\right]$$

1/15×1st, -1/15×2nd, -1/15×3rd

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 4/5 & 4/3 & -1/15 \\ 0 & 1 & 0 & 2/5 & 1/3 & 2/15 \\ 0 & 0 & 1 & -1/5 & -2/5 & -1/15 \end{array}\right]$$

$$\begin{bmatrix} \frac{4}{5} & \frac{4}{3} & -\frac{1}{15} \\ \frac{2}{5} & \frac{1}{3} & \frac{2}{15} \\ -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{15} \end{bmatrix}$$

Use an Inverse to Solve System of Equations

- Write system as matrices
- $AX = B$ (coefficients · variables = constants)
- $A^{-1}AX = A^{-1}B$
- $IX = A^{-1}B$
- $X = A^{-1}B$
- Solve by multiplying the inverse of the coefficients with the constants

Solve $\begin{cases} 2x + 3y = 0 \\ x - 4y = 7 \end{cases}$

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

Find inverse of coefficients

$$\left[\begin{array}{cc:cc} 2 & 3 & 1 & 0 \\ 1 & -4 & 0 & 1 \end{array}\right]$$

1st add to -2×3rd

$$\left[\begin{array}{cc:cc} 2 & 3 & 1 & 0 \\ 0 & 11 & 1 & -2 \end{array}\right]$$

-3×2nd add to 11×1st

$$\left[\begin{array}{cc:cc} 22 & 0 & 8 & 6 \\ 0 & 11 & 1 & -2 \end{array}\right]$$

1/22×1st, 1/11×2nd

$$\left[\begin{array}{cc:cc} 1 & 0 & 4/11 & 3/11 \\ 0 & 1 & 1/11 & -2/11 \end{array}\right]$$

Multiply inverse with constants

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ \frac{1}{11} & -\frac{2}{11} \end{bmatrix} \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{21}{11} \\ -\frac{14}{11} \end{bmatrix}$$

(21/11, -14/11)