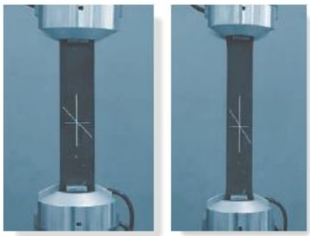
STRAINS

2.1 Deformation

Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as ***deformation***, and they may be either highly visible or practically unnoticeable. For example, a rubber band will undergo a very large deformation when stretched, whereas only slight deformations of structural members occur when a building is occupied by people walking about. Deformation of a body can also occur when the temperature of the body is changed. A typical example is the thermal expansion or contraction of a roof caused by the weather.

In a general sense, the deformation of a body will not be uniform throughout its volume, and so the change in geometry of any line segment within the body may vary substantially along its length. Hence, to study deformational changes in a more uniform manner, we will consider line segments that are very short and located in the neighborhood of a point. Realize, however, that these changes will also depend on the orientation of the line segment at the point. For example, a line segment may elongate if it is oriented in one direction, whereas it may contract if it is oriented in another direction.



Note the before and after positions of three different line segments on this rubber membrane which is subjected to tension. The vertical line is lengthened, the horizontal line is shortened, and the inclined line changes its length and rotates.

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2.2 Strain

In order to describe the deformation of a body by changes in length of line segments and the changes in the angles between them, we will develop the concept of strain. Strain is actually measured by experiments, and once the strain is obtained, it will be shown in the next chapter how it can be related to the stress acting within the body.

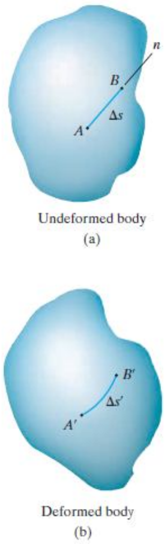

Fig. 2–1

Normal Strain. If we define the normal strain as the change in length of a line per unit length, then we will not have to specify the *actual length* of any particular line segment. Consider, for example, the line AB, which is contained within the undeformed body shown in Fig. 2–1a. This line lies along the n axis and has an original length of Δs. After deformation, points A and B are displaced to A' and B', and the line becomes a curve having a length of $\Delta s'$, Fig. 2–1b. The change in length of the line is therefore $\Delta s' - \Delta s$. If we define the *average normal strain* using the symbol ϵ_{avg} (epsilon), then

$$\epsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s} \qquad (2\text{–}1)$$

As point B is chosen closer and closer to point A, the length of the line will become shorter and shorter, such that $\Delta s \to 0$. Also, this causes B' to approach A', such that $\Delta s' \to 0$. Consequently, in the limit the normal strain at point A and in the direction of n is

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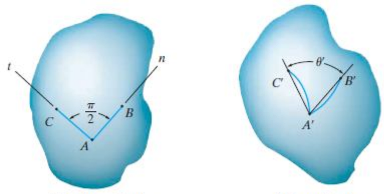
Shear Strain. Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we select two line segments that are originally perpendicular to one another, then the change in angle that occurs between these two line segments is referred to as ***shear strain***. This angle is denoted by γ (gamma) and is always measured in radians (rad), which are dimensionless. For example, consider the line segments AB and AC originating from the same point A in a body, and directed along the perpendicular n and t axes, Fig. 2–2a. After deformation, the ends of both lines are displaced, and the lines themselves become curves, such that the angle between them at A is θ', Fig. 2–2b. Hence the shear strain at point A associated with the n and t axes becomes

$$\gamma_{nt} = \frac{\pi}{2} - \lim_{\substack{B \to A \text{ along } n \\ C \to A \text{ along } t}} \theta' \qquad (2\text{–}3)$$



Undeformed body (a) Deformed body (b)

Fig. 2–2

Notice that if θ' is smaller than $\pi/2$ the shear strain is positive, whereas if θ' is larger than $\pi/2$ the shear strain is negative.

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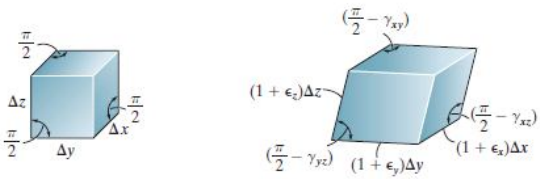
Notice that the ***normal strains cause a change in volume*** of the element, whereas the ***shear strains cause a change in its shape***. Of course, both of these effects occur simultaneously during the deformation.

In summary, then, the *state of strain* at a point in a body requires specifying three normal strains, ϵ_x, ϵ_y, ϵ_z, and three shear strains, γ_{xy}, γ_{yz}, γ_{xz}. These strains completely describe the deformation of a rectangular volume element of material located at the point and oriented so that its sides are originally parallel to the x, y, z axes. Provided these strains are defined at all points in the body, then the deformed shape of the body can be determined.



Undeformed element (b) Deformed element (c)

Fig. 2–3

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Important Points

- Loads will cause all material bodies to deform and, as a result, points in a body will undergo *displacements or changes in position.*
- *Normal strain* is a measure per unit length of the elongation or contraction of a small line segment in the body, whereas *shear strain* is a measure of the change in angle that occurs between two small line segments that are originally perpendicular to one another.
- The state of strain at a point is characterized by six strain components: three normal strains ϵ_x, ϵ_y, ϵ_z and three shear strains γ_{xy}, γ_{yz}, γ_{xz}. These components depend upon the original orientation of the line segments and their location in the body.
- Strain is the geometrical quantity that is measured using experimental techniques. Once obtained, the stress in the body can then be determined from material property relations, as discussed in the next chapter.
- Most engineering materials undergo very small deformations, and so the normal strain $\epsilon \ll 1$. This assumption of "small strain analysis" allows the calculations for normal strain to be simplified, since first-order approximations can be made about their size.

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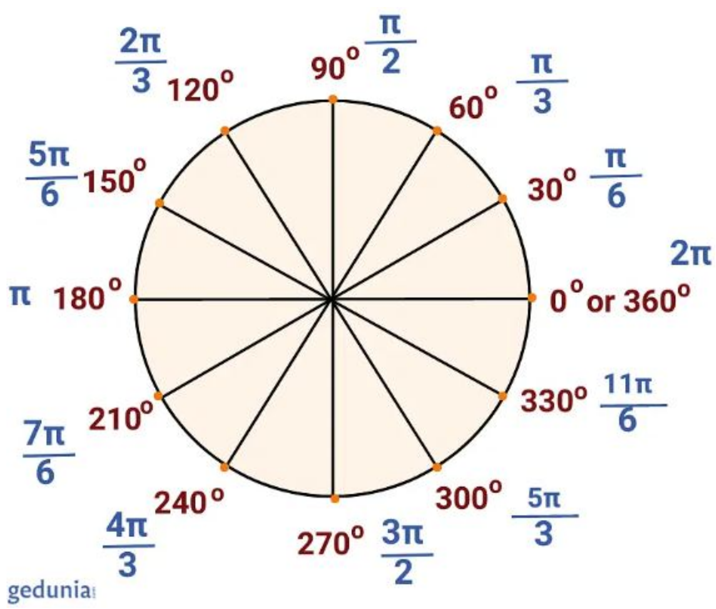

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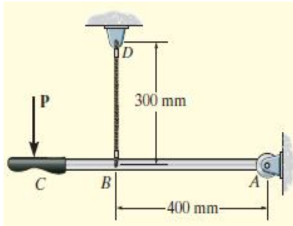
Example

Force P is applied to the arm ABC, the arm rotates about pin A through an angle of 0.05°
Determine the normal strain in wire BD.

Solution



$$\alpha = \tan^{-1}\left(\frac{400\text{ mm}}{300\text{ mm}}\right) = 53.1301°$$

Then

$$\phi = 90° - \alpha + 0.05° = 90° - 53.1301° + 0.05° = 36.92°$$

For triangle ABD the Pythagorean theorem gives

$$L_{AD} = \sqrt{(300\text{ mm})^2 + (400\text{ mm})^2} = 500\text{ mm}$$

Using this result and applying the law of cosines to triangle $AB'D$,

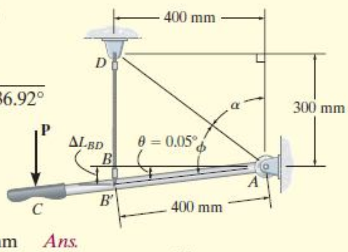
$$L_{B'D} = \sqrt{L_{AD}^2 + L_{AB'}^2 - 2(L_{AD})(L_{AB'})\cos\phi}$$

$$= \sqrt{(500\text{ mm})^2 + (400\text{ mm})^2 - 2(500\text{ mm})(400\text{ mm})\cos 36.92°}$$

$$= 300.3491\text{ mm}$$

Normal Strain.

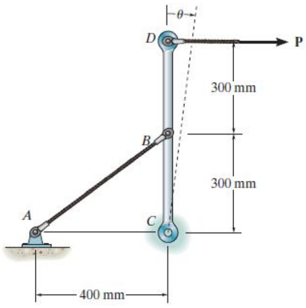
$$\epsilon_{BD} = \frac{L_{B'D} - L_{BD}}{L_{BD}} = \frac{300.3491\text{ mm} - 300\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \quad \textit{Ans.}$$



(b)

Example

If Force P is applied to the end D and cause rotate member CBD by ϴ= 0.3°
Determine the normal strain in the cable AB.



Solution

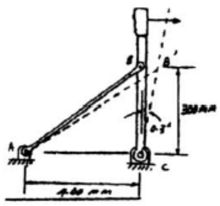
$$AB = \sqrt{400^2 + 300^2} = 500\text{ mm}$$

$$AB' = \sqrt{400^2 + 300^2 - 2(400)(300)\cos 90.3°}$$

$$= 501.255\text{ mm}$$

$$\epsilon_{AB} = \frac{AB' - AB}{AB} = \frac{501.255 - 500}{500}$$

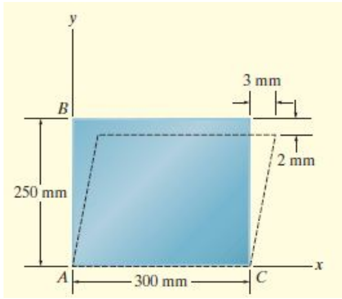
$$= 0.00251\text{ mm/mm}$$



Example

The plate is deformed into the dashed shape shown. Determine :

a. The average normal strain along the side AB.
b. The average shear strain at A.



Solution

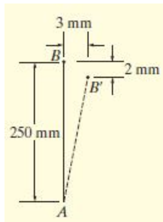
Part (a). Line AB, coincident with the y axis, becomes line AB' after deformation, as shown in Fig. 2–6b. The length of AB' is

$$AB' = \sqrt{(250\text{ mm} - 2\text{ mm})^2 + (3\text{ mm})^2} = 248.018\text{ mm}$$

The average normal strain for AB is therefore

$$(\epsilon_{AB})_{\text{avg}} = \frac{AB' - AB}{AB} = \frac{248.018\text{ mm} - 250\text{ mm}}{250\text{ mm}}$$

$$= -7.93(10^{-3})\text{ mm/mm}$$



Part (b). As noted in Fig. 2–6c, the once 90° angle BAC between the sides of the plate at A changes to θ' due to the displacement of B to B'. Since $\gamma_{xy} = \pi/2 - \theta'$, then γ_{xy} is the angle shown in the figure. Thus,

$$\gamma_{xy} = \tan^{-1}\left(\frac{3\text{ mm}}{250\text{ mm} - 2\text{ mm}}\right) = 0.0121\text{ rad} \qquad \textit{Ans.}$$

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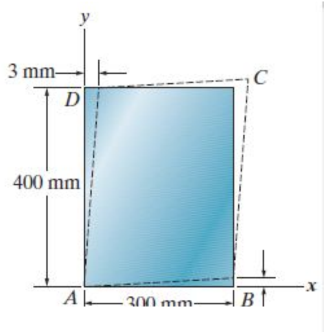
Example

The piece of rubber is originally rectangular and subjected to the deformation shown. Determine the average shear strain ϒxy at A.



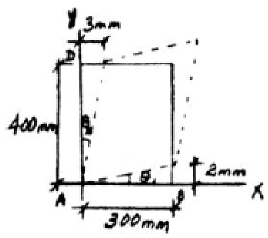
Solution

$$\theta_1 = \tan\theta_1 = \frac{2}{300} = 0.006667\text{ rad}$$

$$\theta_2 = \tan\theta_2 = \frac{3}{400} = 0.0075\text{ rad}$$

$$\gamma_{xy} = \theta_1 + \theta_2$$

$$= 0.006667 + 0.0075 = 0.0142\text{ rad}$$



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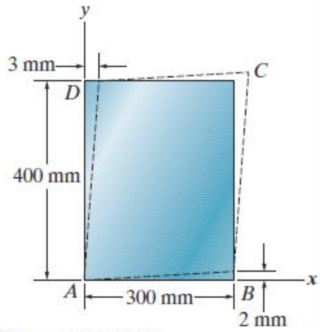
Example

The piece of rubber is originally rectangular and subjected to the deformation shown. Determine the average normal strain along the diagonal DB and side AD

Solution



$$AD' = \sqrt{(400)^2 + (3)^2} = 400.01125 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{3}{400}\right) = 0.42971°$$

$$AB' = \sqrt{(300)^2 + (2)^2} = 300.00667$$

$$\varphi = \tan^{-1}\left(\frac{2}{300}\right) = 0.381966°$$

$$\alpha = 90° - 0.42971° - 0.381966° = 89.18832°$$

$$D'B' = \sqrt{(400.01125)^2 + (300.00667)^2 - 2(400.01125)(300.00667)\cos(89.18832°)}$$

$$D'B' = 496.6014 \text{ mm}$$

$$DB = \sqrt{(300)^2 + (400)^2} = 500 \text{ mm}$$

$$\epsilon_{DB} = \frac{496.6014 - 500}{500} = -0.00680 \text{ mm/mm} \qquad \textbf{Ans.}$$

$$\epsilon_{AD} = \frac{400.01125 - 400}{400} = 0.0281(10^{-3}) \text{ mm/mm} \qquad \textbf{Ans.}$$

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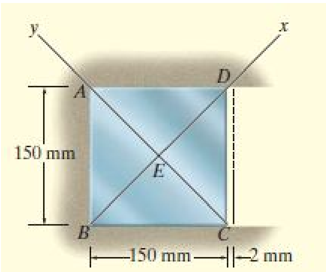
Example

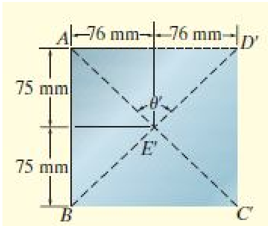
The plate shown is fixed at AB and held in the horizontal guides at its top and bottom AD and BC. If the right side CD displaced 2mm.

Determine:

a. the average normal strain along the diagonal AC.
b. The shear strain at E.

Solution





$$AC = \sqrt{(0.150 \text{ m})^2 + (0.150 \text{ m})^2} = 0.21213 \text{ m}$$

$$AC' = \sqrt{(0.150 \text{ m})^2 + (0.152 \text{ m})^2} = 0.21355 \text{ m}$$

Therefore the average normal strain along the diagonal is

$$(\epsilon_{AC})_{\text{avg}} = \frac{AC' - AC}{AC} = \frac{0.21355 \text{ m} - 0.21213 \text{ m}}{0.21213 \text{ m}}$$

$$= 0.00669 \text{ mm/mm}$$

$$\tan\left(\frac{\theta'}{2}\right) = \frac{76 \text{ mm}}{75 \text{ mm}}$$

$$\theta' = 90.759° = \left(\frac{\pi}{180°}\right)(90.759°) = 1.58404 \text{ rad}$$

Applying Eq. 2–3, the shear strain at E is therefore

$$\gamma_{xy} = \frac{\pi}{2} - 1.58404 \text{ rad} = -0.0132 \text{ rad}$$

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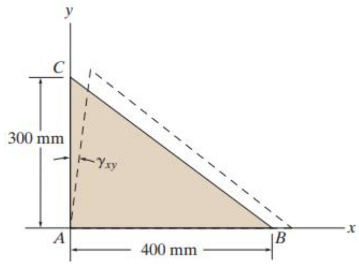
***2–16.** The triangular plate ABC is deformed into the shape shown by the dashed lines. If at A, $\varepsilon_{AB} = 0.0075$, $\epsilon_{AC} = 0.01$ and $\gamma_{xy} = 0.005$ rad, determine the average normal strain along edge BC.



Average Normal Strain: The stretched length of sides AB and AC are

$$L_{AC'} = (1 + \varepsilon_y)L_{AC} = (1 + 0.01)(300) = 303 \text{ mm}$$

$$L_{AB'} = (1 + \varepsilon_x)L_{AB} = (1 + 0.0075)(400) = 403 \text{ mm}$$

Also,

$$\theta = \frac{\pi}{2} - 0.005 = 1.5658 \text{ rad}\left(\frac{180°}{\pi \text{ rad}}\right) = 89.7135°$$

The unstretched length of edge BC is

$$L_{BC} = \sqrt{300^2 + 400^2} = 500 \text{ mm}$$

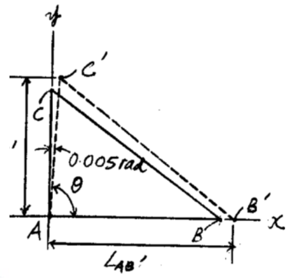
and the stretched length of this edge is

$$L_{B'C'} = \sqrt{303^2 + 403^2 - 2(303)(403)\cos 89.7135°}$$

$$= 502.9880 \text{ mm}$$

We obtain,

$$\epsilon_{BC} = \frac{L_{B'C'} - L_{BC}}{L_{BC}} = \frac{502.9880 - 500}{500} = 5.98(10^{-3}) \text{ mm/mm}$$ **Ans.**



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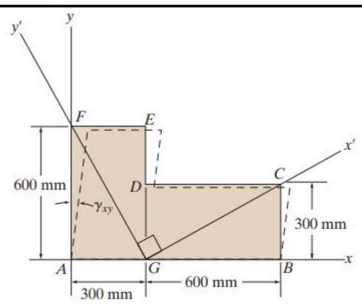
2–17. The plate is deformed uniformly into the shape shown by the dashed lines. If at A, $\gamma_{xy} = 0.0075$ rad., while $\epsilon_{AB} = \epsilon_{AF} = 0$, determine the average shear strain at point G with respect to the x' and y' axes.



Geometry: Here, $\gamma_{xy} = 0.0075 \text{ rad}\left(\frac{180°}{\pi \text{ rad}}\right) = 0.4297°$. Thus,

$$\psi = 90° - 0.4297° = 89.5703° \qquad \beta = 90° + 0.4297° = 90.4297°$$

Subsequently, applying the cosine law to triangles AGF' and GBC', Fig. *a*,

$$L_{GF'} = \sqrt{600^2 + 300^2 - 2(600)(300)\cos 89.5703°} = 668.8049 \text{ mm}$$

$$L_{GC'} = \sqrt{600^2 + 300^2 - 2(600)(300)\cos 90.4297°} = 672.8298 \text{ mm}$$

Then, applying the sine law to the same triangles,

$$\frac{\sin\phi}{600} = \frac{\sin 89.5703°}{668.8049}; \qquad \phi = 63.7791°$$

$$\frac{\sin\alpha}{300} = \frac{\sin 90.4297°}{672.8298}; \qquad \alpha = 26.4787°$$

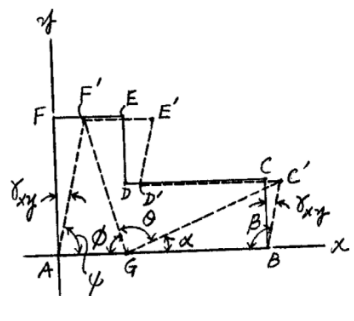
Thus,

$$\theta = 180° - \phi - \alpha = 180° - 63.7791° - 26.4787°$$

$$= 89.7422°\left(\frac{\pi \text{ rad}}{180°}\right) = 1.5663 \text{ rad}$$

Shear Strain:

$$(\gamma_G)_{x'y'} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - 1.5663 = 4.50(10^{-3}) \text{ rad}$$ **Ans.**



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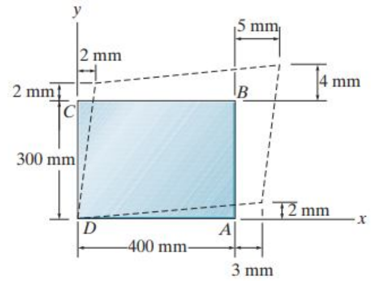
2–18. The piece of plastic is originally rectangular. Determine the shear strain γ_{xy} at corners A and B if the plastic distorts as shown by the dashed lines.

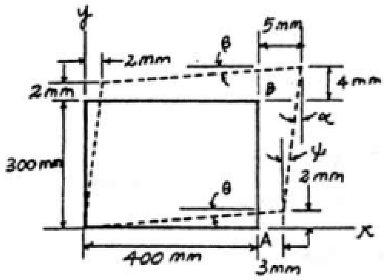


Geometry: For small angles,

$$\alpha = \psi = \frac{2}{302} = 0.00662252 \text{ rad}$$

$$\beta = \theta = \frac{2}{403} = 0.00496278 \text{ rad}$$

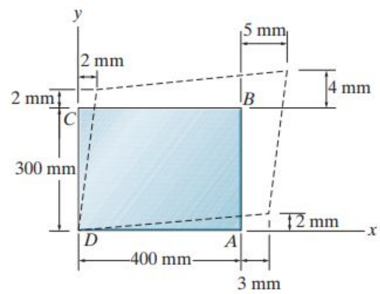
Shear Strain:



$$(\gamma_B)_{xy} = \alpha + \beta$$
$$= 0.0116 \text{ rad} = 11.6\left(10^{-3}\right) \text{ rad}$$

$$(\gamma_A)_{xy} = \theta + \psi$$
$$= 0.0116 \text{ rad} = 11.6\left(10^{-3}\right) \text{ rad}$$

***2–20.** The piece of plastic is originally rectangular. Determine the average normal strain that occurs along the diagonals AC and DB.



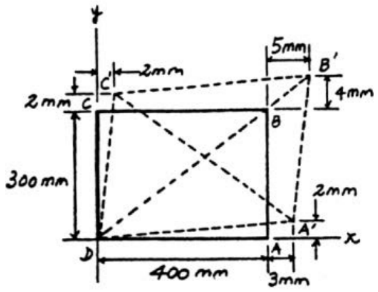
Geometry:

$$AC = DB = \sqrt{400^2 + 300^2} = 500 \text{ mm}$$

$$DB' = \sqrt{405^2 + 304^2} = 506.4 \text{ mm}$$

$$A'C' = \sqrt{401^2 + 300^2} = 500.8 \text{ mm}$$

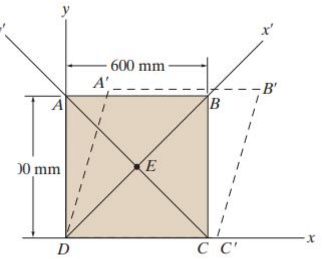
Average Normal Strain:



$$\epsilon_{AC} = \frac{A'C' - AC}{AC} = \frac{500.8 - 500}{500}$$
$$= 0.00160 \text{ mm/mm} = 1.60\left(10^{-3}\right) \text{ mm/mm}$$

$$\epsilon_{DB} = \frac{DB' - DB}{DB} = \frac{506.4 - 500}{500}$$
$$= 0.0128 \text{ mm/mm} = 12.8\left(10^{-3}\right) \text{ mm/mm}$$

2–26. The square plate is deformed into the shape shown by the dashed lines. If DC has a normal strain $\epsilon_x = 0.004$, DA has a normal strain $\epsilon_y = 0.005$ and at D, $\gamma_{xy} = 0.02$ rad, determine the average normal strain along diagonal CA.



Average Normal Strain: The stretched length of sides DA and DC are

$$L_{DC'} = (1 + \epsilon_x)L_{DC} = (1 + 0.004)(600) = 602.4 \text{ mm}$$

$$L_{DA'} = (1 + \epsilon_y)L_{DA} = (1 + 0.005)(600) = 603 \text{ mm}$$

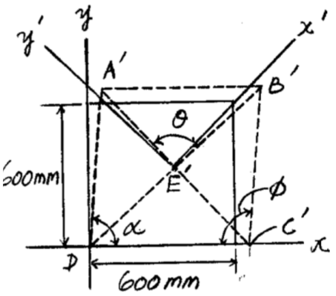
Also,

$$\alpha = \frac{\pi}{2} - 0.02 = 1.5508 \text{ rad}\left(\frac{180^\circ}{\pi \text{ rad}}\right) = 88.854^\circ$$

Thus, the length of $C'A'$ can be determined using the cosine law with reference to Fig. *a*.

$$L_{C'A'} = \sqrt{602.4^2 + 603^2 - 2(602.4)(603)\cos 88.854^\circ}$$

$$= 843.7807 \text{ mm}$$



The original length of diagonal CA can be determined using Pythagorean's theorem.

$$L_{CA} = \sqrt{600^2 + 600^2} = 848.5281 \text{ mm}$$

Thus,

$$(\epsilon_{avg})_{CA} = \frac{L_{C'A'} - L_{CA}}{L_{CA}} = \frac{843.7807 - 848.5281}{848.5281} = -5.59(10^{-3}) \text{ mm/mm} \quad \textbf{Ans.}$$

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