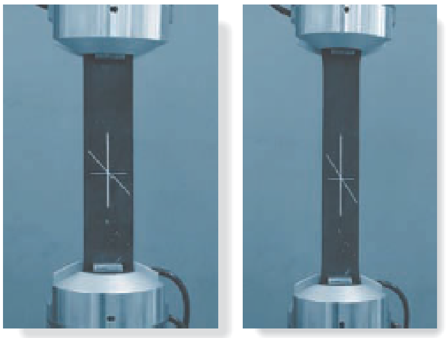
STRAINS

2.1 Deformation

Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as ***deformation***, and they may be either highly visible or practically unnoticeable. For example, a rubber band will undergo a very large deformation when stretched, whereas only slight deformations of structural members occur when a building is occupied by people walking about. Deformation of a body can also occur when the temperature of the body is changed. A typical example is the thermal expansion or contraction of a roof caused by the weather.

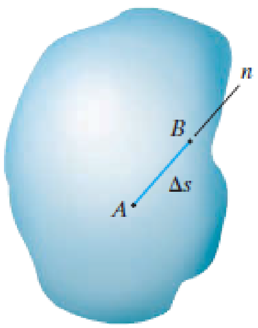
In a general sense, the deformation of a body will not be uniform throughout its volume, and so the change in geometry of any line segment within the body may vary substantially along its length. Hence, to study deformational changes in a more uniform manner, we will consider line segments that are very short and located in the neighborhood of a point. Realize, however, that these changes will also depend on the orientation of the line segment at the point. For example, a line segment may elongate if it is oriented in one direction, whereas it may contract if it is oriented in another direction.



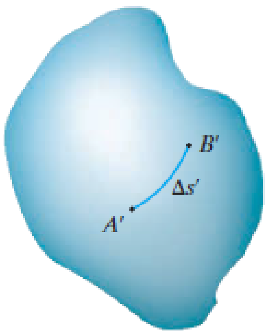
Note the before and after positions of three different line segments on this rubber membrane which is subjected to tension. The vertical line is lengthened, the horizontal line is shortened, and the inclined line changes its length and rotates.

2.2 Strain

In order to describe the deformation of a body by changes in length of line segments and the changes in the angles between them, we will develop the concept of strain. Strain is actually measured by experiments, and once the strain is obtained, it will be shown in the next chapter how it can be related to the stress acting within the body.



Undeformed body
(a)



Deformed body
(b)

Fig. 2–1

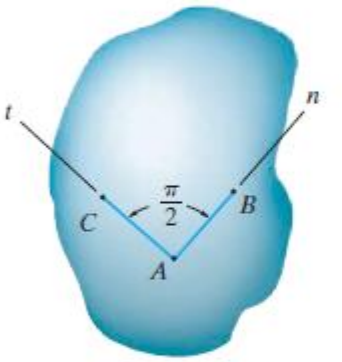
Normal Strain. If we define the normal strain as the change in length of a line per unit length, then we will not have to specify the *actual length* of any particular line segment. Consider, for example, the line AB, which is contained within the undeformed body shown in Fig. 2–1*a*. This line lies along the n axis and has an original length of Δs. After deformation, points A and B are displaced to A' and B', and the line becomes a curve having a length of $\Delta s'$, Fig. 2–1*b*. The change in length of the line is therefore $\Delta s' - \Delta s$. If we define the *average normal strain* using the symbol ϵ_{avg} (epsilon), then

$$\epsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s} \qquad (2\text{–}1)$$

As point B is chosen closer and closer to point A, the length of the line will become shorter and shorter, such that $\Delta s \rightarrow 0$. Also, this causes B' to approach A', such that $\Delta s' \rightarrow 0$. Consequently, in the limit the normal strain at point A and in the direction of n is

Shear Strain. Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we select two line segments that are originally perpendicular to one another, then the change in angle that occurs between these two line segments is referred to as ***shear strain***. This angle is denoted by γ (gamma) and is always measured in radians (rad), which are dimensionless. For example, consider the line segments AB and AC originating from the same point A in a body, and directed along the perpendicular n and t axes, Fig. 2–2*a*. After deformation, the ends of both lines are displaced, and the lines themselves become curves, such that the angle between them at A is θ', Fig. 2–2*b*. Hence the shear strain at point A associated with the n and t axes becomes

$$\gamma_{nt} = \frac{\pi}{2} - \lim_{\substack{B \to A \text{ along } n \\ C \to A \text{ along } t}} \theta' \qquad (2–3)$$

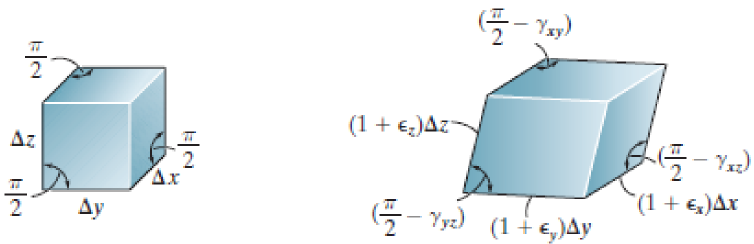


Undeformed body
(a)

Notice that if θ' is smaller than $\pi/2$ the shear strain is positive, whereas if θ' is larger than $\pi/2$ the shear strain is negative.

Notice that the ***normal strains cause a change in volume*** of the element, whereas the ***shear strains cause a change in its shape***. Of course, both of these effects occur simultaneously during the deformation.

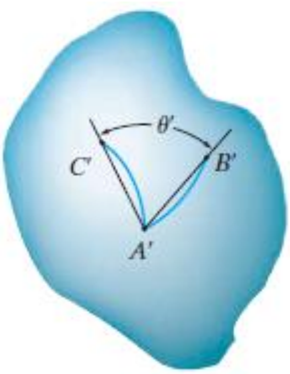
In summary, then, the *state of strain* at a point in a body requires specifying three normal strains, ϵ_x, ϵ_y, ϵ_z, and three shear strains, γ_{xy}, γ_{yz}, γ_{xz}. These strains completely describe the deformation of a rectangular volume element of material located at the point and oriented so that its sides are originally parallel to the x, y, z axes. Provided these strains are defined at all points in the body, then the deformed shape of the body can be determined.



Undeformed element
(b)

Deformed element
(c)

Fig. 2–3

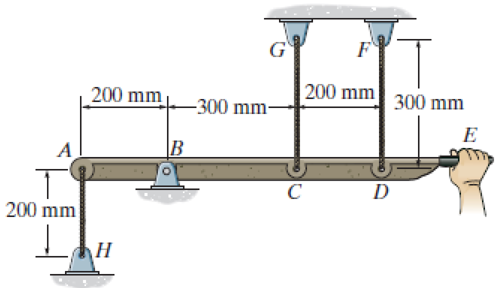


Deformed body
(b)

Important Points

- Loads will cause all material bodies to deform and, as a result, points in a body will undergo *displacements or changes in position.*
- *Normal strain* is a measure per unit length of the elongation or contraction of a small line segment in the body, whereas *shear strain* is a measure of the change in angle that occurs between two small line segments that are originally perpendicular to one another.
- The state of strain at a point is characterized by six strain components: three normal strains ϵ_x, ϵ_y, ϵ_z and three shear strains γ_{xy}, γ_{yz}, γ_{xz}. These components depend upon the original orientation of the line segments and their location in the body.
- Strain is the geometrical quantity that is measured using experimental techniques. Once obtained, the stress in the body can then be determined from material property relations, as discussed in the next chapter.
- Most engineering materials undergo very small deformations, and so the normal strain $\epsilon \ll 1$. This assumption of "small strain analysis" allows the calculations for normal strain to be simplified, since first-order approximations can be made about their size.

***2–4.** The force applied at the handle of the rigid lever causes the lever to rotate clockwise about the pin B through an angle of 2°. Determine the average normal strain developed in each wire. The wires are unstretched when the lever is in the horizontal position.



Geometry: The lever arm rotates through an angle of $\theta = \left(\frac{2°}{180}\right)\pi \text{ rad} = 0.03491 \text{ rad}$.

Since θ is small, the displacements of points A, C, and D can be approximated by

$$\delta_A = 200(0.03491) = 6.9813 \text{ mm}$$

$$\delta_C = 300(0.03491) = 10.4720 \text{ mm}$$

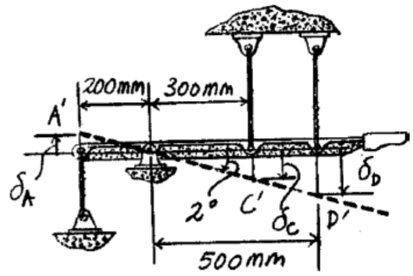
$$\delta_D = 500(0.03491) = 17.4533 \text{ mm}$$

Average Normal Strain: The unstretched length of wires AH, CG, and DF are $L_{AH} = 200$ mm, $L_{CG} = 300$ mm, and $L_{DF} = 300$ mm. We obtain

$$(\epsilon_{avg})_{AH} = \frac{\delta_A}{L_{AH}} = \frac{6.9813}{200} = 0.0349 \text{ mm/mm} \qquad \textbf{Ans.}$$

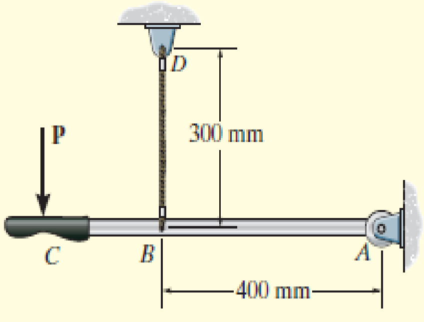
$$(\epsilon_{avg})_{CG} = \frac{\delta_C}{L_{CG}} = \frac{10.4720}{300} = 0.0349 \text{ mm/mm} \qquad \textbf{Ans.}$$

$$(\epsilon_{avg})_{DF} = \frac{\delta_D}{L_{DF}} = \frac{17.4533}{300} = 0.0582 \text{ mm/mm} \qquad \textbf{Ans.}$$



EXAMPLE 2.2

When force **P** is applied to the rigid lever arm *ABC* in Fig. 2–5*a*, the arm rotates counterclockwise about pin *A* through an angle of 0.05°. Determine the normal strain developed in wire *BD*.



(a)

SOLUTION I

Geometry. The orientation of the lever arm after it rotates about point *A* is shown in Fig. 2–5*b*. From the geometry of this figure,

$$\alpha = \tan^{-1}\left(\frac{400 \text{ mm}}{300 \text{ mm}}\right) = 53.1301°$$

Then

$$\phi = 90° - \alpha + 0.05° = 90° - 53.1301° + 0.05° = 36.92°$$

For triangle *ABD* the Pythagorean theorem gives

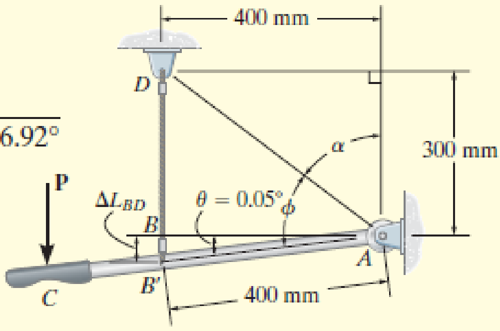
$$L_{AD} = \sqrt{(300 \text{ mm})^2 + (400 \text{ mm})^2} = 500 \text{ mm}$$

Using this result and applying the law of cosines to triangle *AB′D*,

$$L_{B'D} = \sqrt{L_{AD}^2 + L_{AB'}^2 - 2(L_{AD})(L_{AB'})\cos\phi}$$

$$= \sqrt{(500 \text{ mm})^2 + (400 \text{ mm})^2 - 2(500 \text{ mm})(400 \text{ mm})\cos 36.92°}$$

$$= 300.3491 \text{ mm}$$



(b)

Fig. 2–5

Normal Strain.

$$\epsilon_{BD} = \frac{L_{B'D} - L_{BD}}{L_{BD}} = \frac{300.3491 \text{ mm} - 300 \text{ mm}}{300 \text{ mm}} = 0.00116 \text{ mm/mm} \qquad \textit{Ans.}$$

SOLUTION II

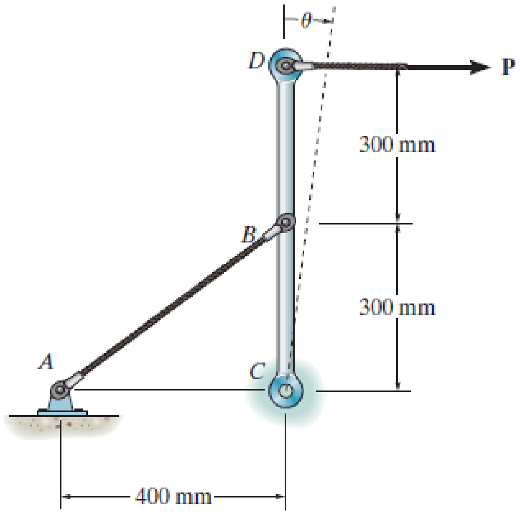
Since the strain is small, this same result can be obtained by approximating the elongation of wire *BD* as ΔL_{BD}, shown in Fig. 2–5*b*. Here,

$$\Delta L_{BD} = \theta L_{AB} = \left[\left(\frac{0.05°}{180°}\right)(\pi \text{ rad})\right](400 \text{ mm}) = 0.3491 \text{ mm}$$

Therefore,

$$\epsilon_{BD} = \frac{\Delta L_{BD}}{L_{BD}} = \frac{0.3491 \text{ mm}}{300 \text{ mm}} = 0.00116 \text{ mm/mm} \qquad \textit{Ans.}$$

***2–8.** Part of a control linkage for an airplane consists of a rigid member *CBD* and a flexible cable *AB*. If a force is applied to the end *D* of the member and causes it to rotate by $\theta = 0.3°$, determine the normal strain in the cable. Originally the cable is unstretched.



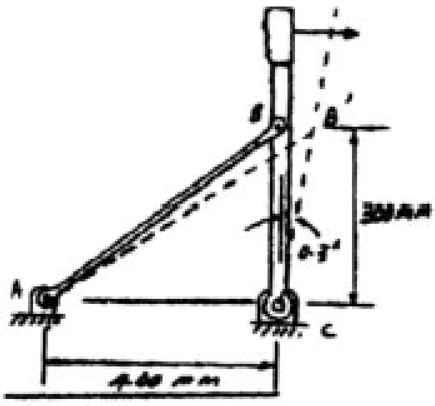
$$AB = \sqrt{400^2 + 300^2} = 500 \text{ mm}$$

$$AB' = \sqrt{400^2 + 300^2 - 2(400)(300)\cos 90.3°}$$

$$= 501.255 \text{ mm}$$

$$\epsilon_{AB} = \frac{AB' - AB}{AB} = \frac{501.255 - 500}{500}$$

$$= 0.00251 \text{ mm/mm}$$



EXAMPLE 2.3

Due to a loading, the plate is deformed into the dashed shape shown in Fig. 2–6*a*. Determine (a) the average normal strain along the side *AB*, and (b) the average shear strain in the plate at *A* relative to the *x* and *y* axes.

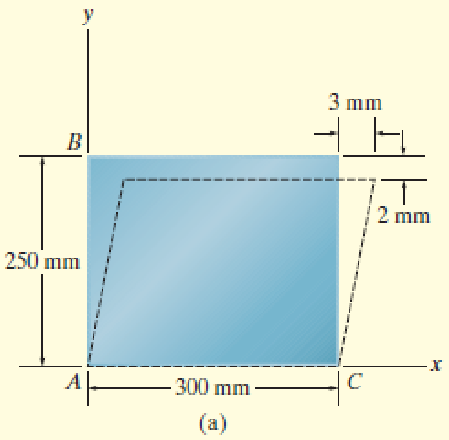
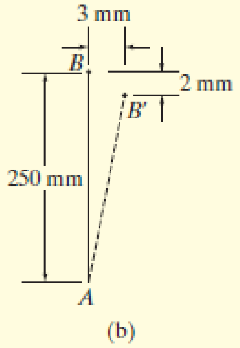


Fig. 2–6



SOLUTION

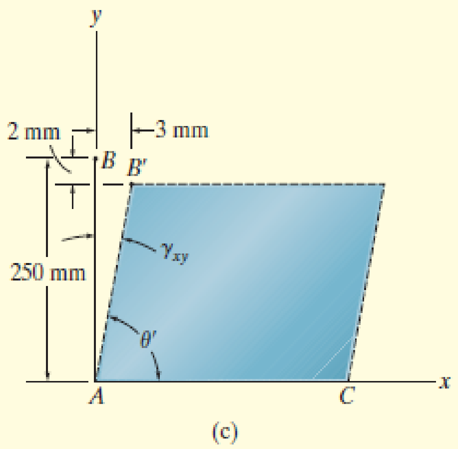
Part (a). Line *AB*, coincident with the *y* axis, becomes line *AB′* after deformation, as shown in Fig. 2–6*b*. The length of *AB′* is

$$AB' = \sqrt{(250\text{ mm} - 2\text{ mm})^2 + (3\text{ mm})^2} = 248.018\text{ mm}$$

The average normal strain for *AB* is therefore

$$(\epsilon_{AB})_{avg} = \frac{AB' - AB}{AB} = \frac{248.018\text{ mm} - 250\text{ mm}}{250\text{ mm}}$$

$$= -7.93(10^{-3})\text{ mm/mm} \qquad \textit{Ans.}$$

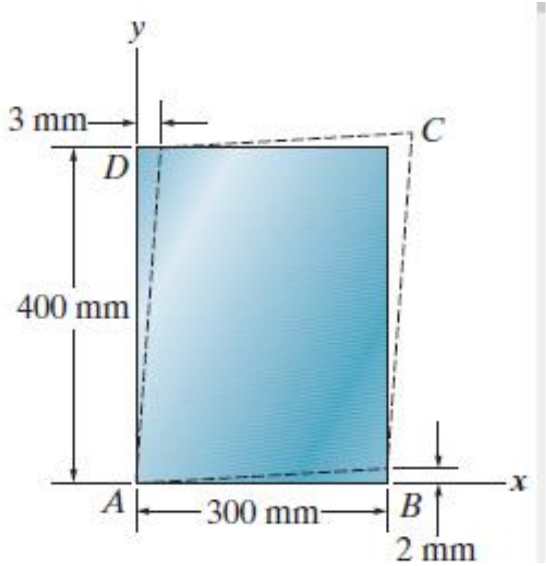


The negative sign indicates the strain causes a contraction of *AB*.

Part (b). As noted in Fig. 2–6*c*, the once 90° angle *BAC* between the sides of the plate at *A* changes to θ' due to the displacement of *B* to *B′*. Since $\gamma_{xy} = \pi/2 - \theta'$, then γ_{xy} is the angle shown in the figure. Thus,

$$\gamma_{xy} = \tan^{-1}\left(\frac{3\text{ mm}}{250\text{ mm} - 2\text{ mm}}\right) = 0.0121\text{ rad} \qquad \textit{Ans.}$$

*2–12. The piece of rubber is originally rectangular. Determine the average shear strain γ_{xy} at A if the corners B and D are subjected to the displacements that cause the rubber to distort as shown by the dashed lines.

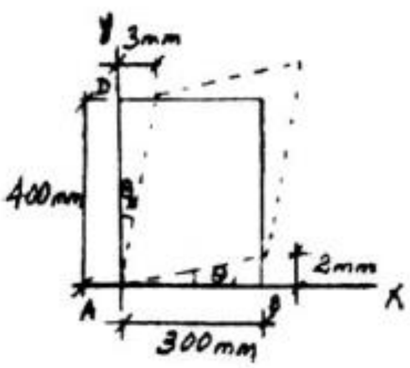


$$\theta_1 = \tan\theta_1 = \frac{2}{300} = 0.006667 \text{ rad}$$

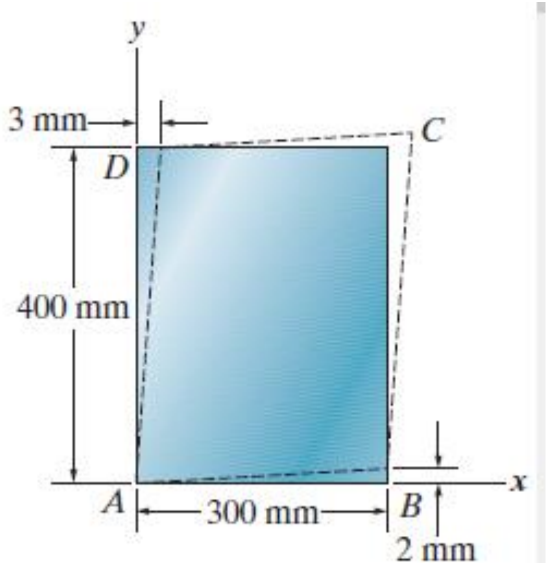
$$\theta_2 = \tan\theta_2 = \frac{3}{400} = 0.0075 \text{ rad}$$

$$\gamma_{xy} = \theta_1 + \theta_2$$

$$= 0.006667 + 0.0075 = 0.0142 \text{ rad}$$



2–13. The piece of rubber is originally rectangular and subjected to the deformation shown by the dashed lines. Determine the average normal strain along the diagonal *DB* and side *AD*.



$$AD' = \sqrt{(400)^2 + (3)^2} = 400.01125 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{3}{400}\right) = 0.42971°$$

$$AB' = \sqrt{(300)^2 + (2)^2} = 300.00667$$

$$\varphi = \tan^{-1}\left(\frac{2}{300}\right) = 0.381966°$$

$$\alpha = 90° - 0.42971° - 0.381966° = 89.18832°$$

$$D'B' = \sqrt{(400.01125)^2 + (300.00667)^2 - 2(400.01125)(300.00667)\cos(89.18832°)}$$

$$D'B' = 496.6014 \text{ mm}$$

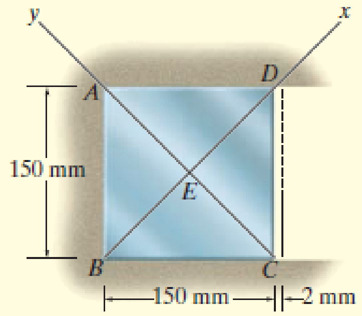
$$DB = \sqrt{(300)^2 + (400)^2} = 500 \text{ mm}$$

$$\epsilon_{DB} = \frac{496.6014 - 500}{500} = -0.00680 \text{ mm/mm} \qquad \textbf{Ans.}$$

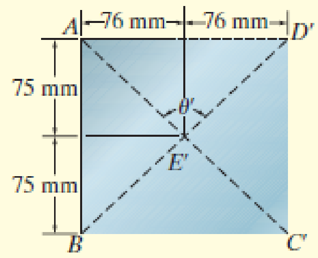
$$\epsilon_{AD} = \frac{400.01125 - 400}{400} = 0.0281(10^{-3}) \text{ mm/mm} \qquad \textbf{Ans.}$$

EXAMPLE 2.4

The plate shown in Fig. 2–7*a* is fixed connected along AB and held in the horizontal guides at its top and bottom, AD and BC. If its right side CD is given a uniform horizontal displacement of 2 mm, determine (a) the average normal strain along the diagonal AC, and (b) the shear strain at E relative to the x, y axes.



(a)



(b)

Fig. 2–7

SOLUTION

Part (a). When the plate is deformed, the diagonal AC becomes AC', Fig. 2–7*b*. The length of diagonals AC and AC' can be found from the Pythagorean theorem. We have

$$AC = \sqrt{(0.150\text{ m})^2 + (0.150\text{ m})^2} = 0.21213\text{ m}$$

$$AC' = \sqrt{(0.150\text{ m})^2 + (0.152\text{ m})^2} = 0.21355\text{ m}$$

Therefore the average normal strain along the diagonal is

$$(\epsilon_{AC})_{avg} = \frac{AC' - AC}{AC} = \frac{0.21355\text{ m} - 0.21213\text{ m}}{0.21213\text{ m}}$$

$$= 0.00669\text{ mm/mm} \qquad \textit{Ans.}$$

Part (b). To find the shear strain at E relative to the x and y axes, it is first necessary to find the angle θ' after deformation, Fig. 2–7*b*. We have

$$\tan\left(\frac{\theta'}{2}\right) = \frac{76\text{ mm}}{75\text{ mm}}$$

$$\theta' = 90.759^\circ = \left(\frac{\pi}{180^\circ}\right)(90.759^\circ) = 1.58404\text{ rad}$$

Applying Eq. 2–3, the shear strain at E is therefore

$$\gamma_{xy} = \frac{\pi}{2} - 1.58404\text{ rad} = -0.0132\text{ rad} \qquad \textit{Ans.}$$

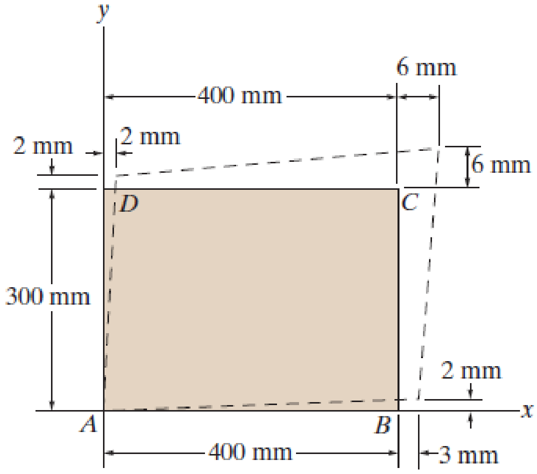
The *negative sign* indicates that the angle θ' is *greater than* 90°.

NOTE: If the x and y axes were horizontal and vertical at point E, then the 90° angle between these axes would not change due to the deformation, and so $\gamma_{xy} = 0$ at point E.

Q:

The rectangular plate is deformed into the shape shown by the dashed lines. Determine:

1. The average **normal strain** along diagonals ***AC*** and ***DB.***
2. The average **shear strain** at corners ***A*** and ***B***

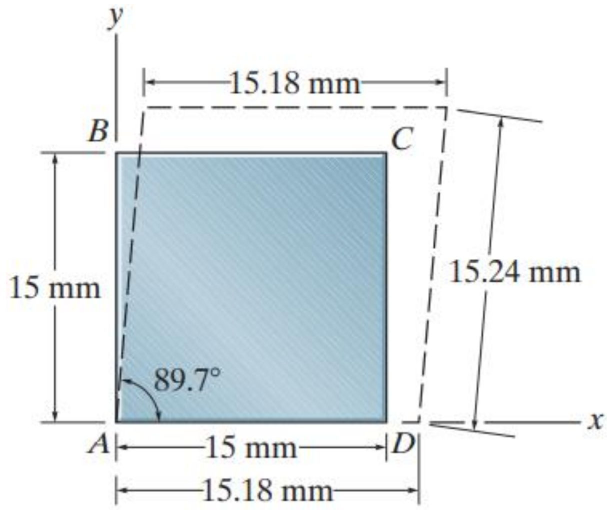


Q:

A square piece of material is deformed into the dashed position. Determine:

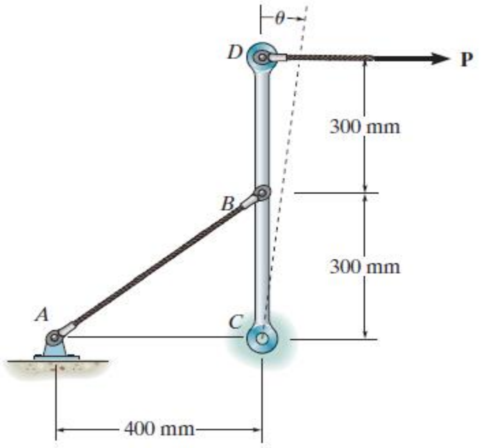
1. The average **normal strain** along diagonals ***AC*** and ***DB.***

The average **shear strain** at corners ***A*** and ***C***



Q:

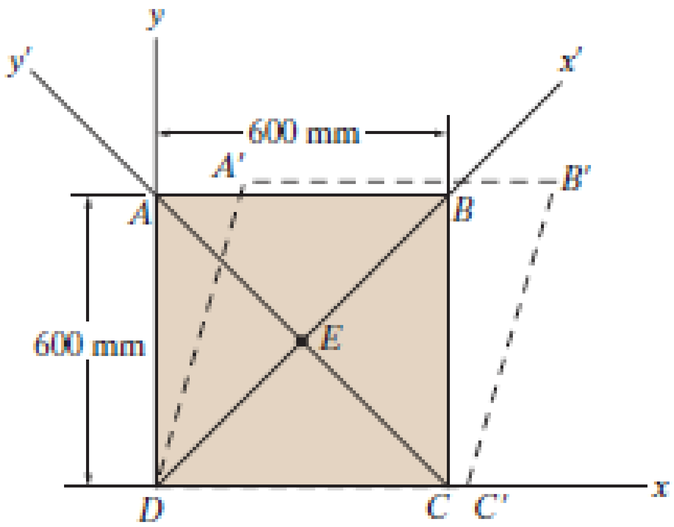
The rigid member *CBD* and flexible cable *AB* is subjected to load at *D*, if the *normal strain* at cable *AB* is *0.0035*, determine the **displacement** of point ***D***.



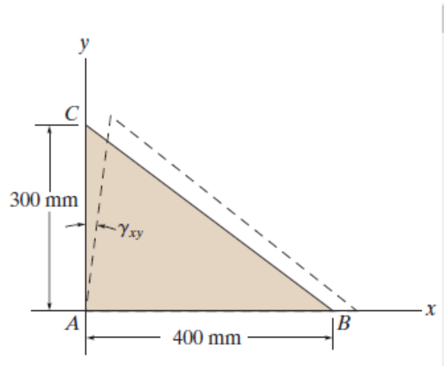
Q:

The square plate is deformed into the shape shown by the dashed lines.

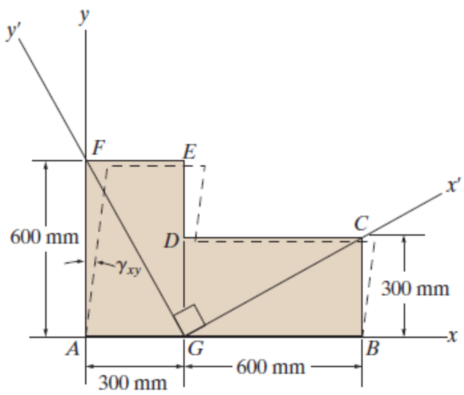
If *DC* has a normal strain $\epsilon_x = 0.004$, *DA* has a normal strain $\epsilon_y = 0.005$ *and* at *D*, $\gamma_{xy} = 0.02$ rad., determine the average **normal strain** along diagonal CA and the average **shear strain** at point ***E*** with respect to the *x'* and *y'* axes.



***2–16.** The triangular plate *ABC* is deformed into the shape shown by the dashed lines. If at *A*, $\varepsilon_{AB} = 0.0075$, $\epsilon_{AC} = 0.01$ and $\gamma_{xy} = 0.005$ rad, determine the average normal strain along edge *BC*.

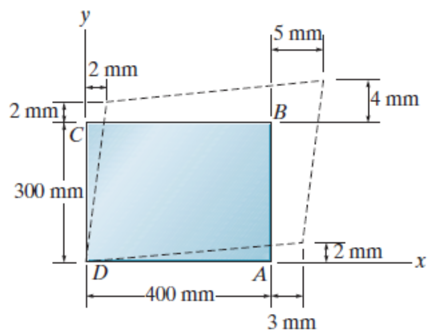


2–17. The plate is deformed uniformly into the shape shown by the dashed lines. If at *A*, $\gamma_{xy} = 0.0075$ rad., while $\epsilon_{AB} = \epsilon_{AF} = 0$, determine the average shear strain at point *G* with respect to the x' and y' axes.



Geometry: Here, $\gamma_{xy} = 0.0075 \text{ rad}\left(\frac{180°}{\pi \text{ rad}}\right) = 0.4297°$. Thus,

2–18. The piece of plastic is originally rectangular. Determine the shear strain γ_{xy} at corners *A* and *B* if the plastic distorts as shown by the dashed lines.



***2–20.** The piece of plastic is originally rectangular. Determine the average normal strain that occurs along the diagonals *AC* and *DB*.

2–21. The rectangular plate is deformed into the shape of a parallelogram shown by the dashed lines. Determine the average shear strain γ_{xy} at corners *A* and *B*.