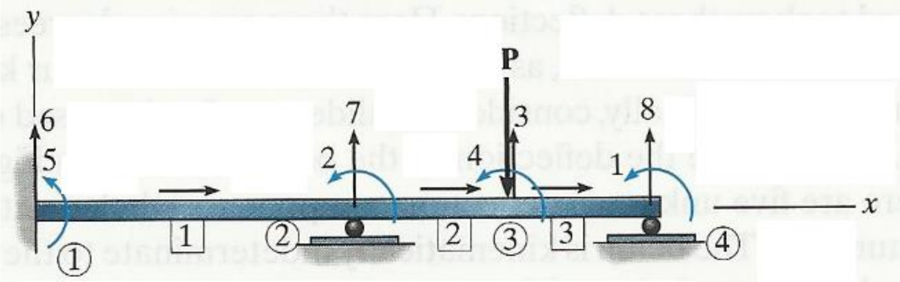
Beam Analysis Using the Stiffness Method

Member and Node Identification. In order to apply the stiffness method to beams, first determine how to subdivide the beam into its component finite elements.

In general, each must be free from load and have a prismatic cross section.

For this reason, the nodes of each element are located at a support or at points where members are connected together, where an external force is applied, where the cross-sectional area suddenly changes, or where the vertical or rotational displacement at a point is to be determined.

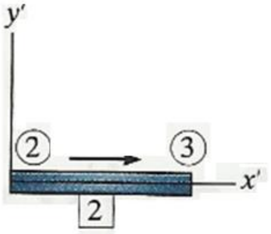
Using the same scheme as that for trusses, four nodes are specified numerically within a circle, and the elements are identified numerically within a square. Also, notice that the "near" and "far" ends of each element are identified by the arrows.



Global and Member Coordinates.

The global coordinate system will be identified using *x, y, z* axes that generally have their origin at a node and are positioned so that the nodes at other points on the beam all have positive coordinates. The local or member *x', y', z'* coordinates have their origin at the "near" end of each element, and the positive *x'* axis is directed towards the "far" end.

In both cases we have used a right-handed coordinate system, so that if the fingers of the right hand are curled from the x (x') axis towards the y (y') axis, the thumb points in the positive direction of the z (z') axis, which is directed out of the page. Notice that for each beam element the x and x' axes will be collinear and the global and member coordinates will all be parallel.



Kinematic Indeterminacy

Once the elements and nodes have been identified, and the global coordinate system has been established, the degrees of freedom for the beam and its kinematic determinacy can be determined.

Consider the effects of both bending and shear, then *each node* on a beam can have two degrees of freedom, namely, a vertical displacement and a rotation.

As in the case of trusses, these linear and rotational displacements will be identified by code numbers.

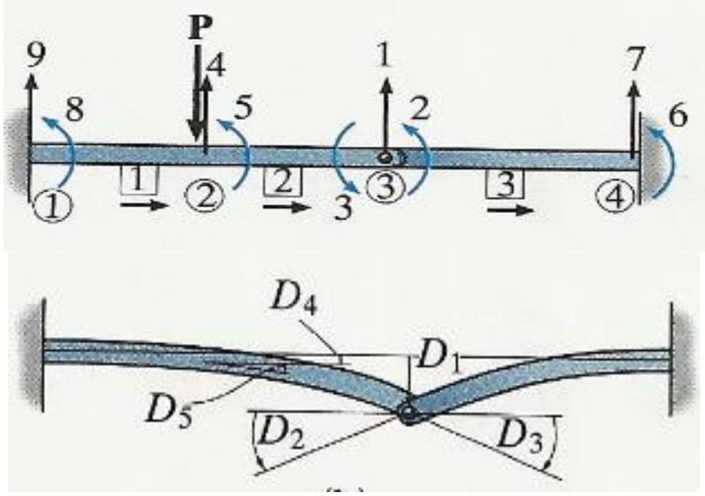
The lowest code numbers will be used to identify the unknown displacements (unconstrained degrees of freedom), and the highest numbers are used to identify the known displacements (constrained degrees of freedom).

Recall that the reason for choosing this method of identification has to do with the convenience of later partitioning the structure stiffness matrix, so that the unknown displacements can be found in the most direct manner.

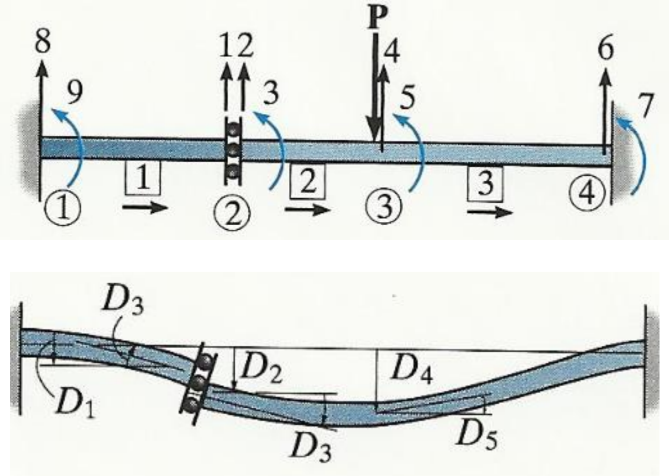
Degree of Freedom

Structure	Nodal DOF	Nodal Force
Plane Truss	Ux, Uy	Fx, Fy
Space Truss	Ux, Uy, Uz	Fx, Fy, Fz
Beam	Uy, θz	Fy, Mz
Plane Frame	Ux, Uy, θz	Fx, Fy, Mz
Space Frame	Ux, Uy, Uz θx, θy, θz	Fx, Fy, Fz, Mx, My, Mz

As example, the beam in Fig. below can be subdivided into three elements and four nodes. In particular, notice that the internal hinge at node 3 deflects the same for both elements 2 and 3; however, the rotation at the end of each element is different. For this reason, three code numbers are used to show these deflections. Here there are nine degrees of freedom, five of which are unknown, and four knowns; again, they are all zero.



Consider the slider mechanism used on the beam in the Fig. below the deflection of the beam is shown, and so there are five unknown deflection components labeled with the lowest code numbers. The beam is kinematically indeterminate to the fifth degree



Development of the stiffness method for beams follows a similar procedure as that used for trusses.

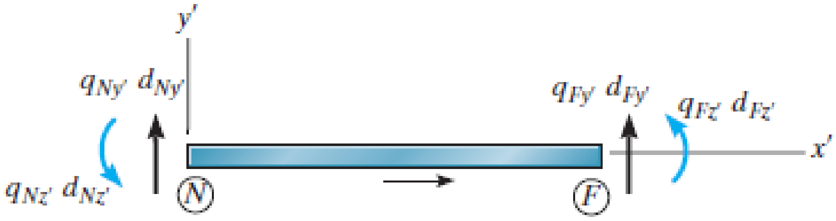
First we must establish the stiffness matrix for each element, and then these matrices are combined to form the beam or structure stiffness matrix.

Using the structure matrix equation, we can then proceed to determine the unknown displacements at the nodes and from this determine the reactions on the beam and the internal shear and moment at the nodes.

Beam-Member Stiffness Matrix

The stiffness matrix for a beam element or member having a constant cross-sectional area and referenced from the local x', y', z' coordinate system.

The origin of the coordinates is placed at the "near" end N, and the positive x' axis extends toward the "far" end F.



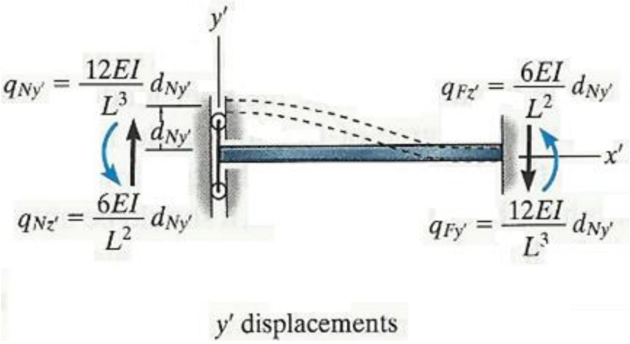
positive sign convention

There are two reactions at each end of the element, consisting of shear forces qNy' and qFy' and bending moments qNz' and qFz': These loadings all act in the positive coordinate directions. In particular, the moments qNz' and qFz' are positive counterclockwise, since by the right-hand rule the moment vectors are then directed along the positive z' axis, which is out of the page.

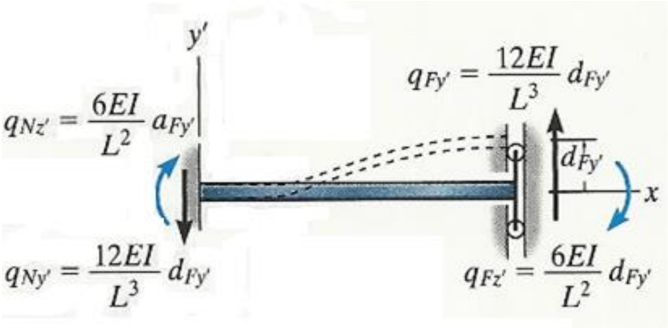
Linear and angular displacements associated with these loadings also follow this same positive sign convention. We will now impose each of these displacements separately and then determine the loadings acting on the member caused by each displacement.

Displacements

When a positive displacement $d_{Ny'}$ is imposed while other possible displacements are prevented, the resulting shear forces and bending moments that are created are shown. When d_Fy' is imposed, the required shear forces and bending moments are shown



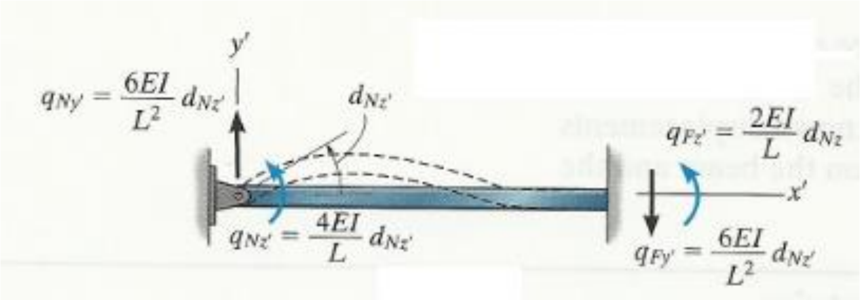
y' displacements

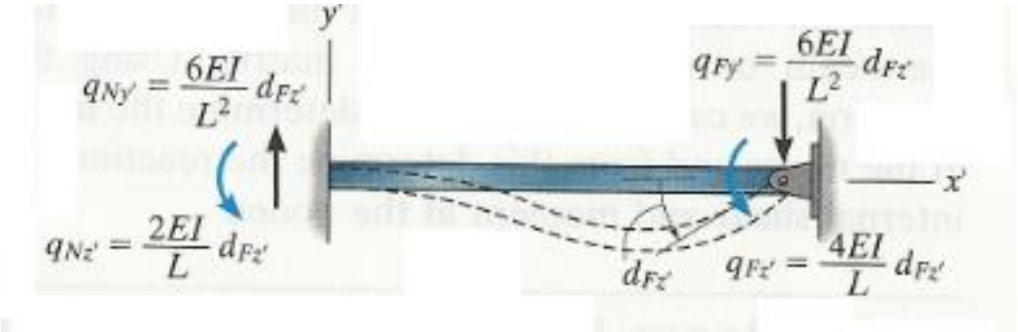


Rotations

If a positive rotation d_{Nz}' is imposed while all other possible displacements are prevented, the required shear forces and moments necessary for the deformation are shown in Fig.

When $d_{Fz'}$ is imposed, the resultant loadings are shown in Fig.





By superposition, if the above results in two previous Figs are added, the resulting four load-displacement relations for the member can be expressed in matrix form as

$$\begin{bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix} = \begin{array}{cccc} N_{y'} & N_{z'} & F_{y'} & F_{z'} \\ \end{array} \left[\begin{array}{cccc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right] \begin{bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix}$$

These equations can also be written in abbreviated form as:

$$\mathbf{q} = \mathbf{kd}$$

The symmetric matrix **k** is referred to as the *member stiffness matrix.*

The **16** influence **coefficients** $\mathbf{k}_{\mathbf{ij}}$ that comprise it account for the **shear-force** and **bending-moment** displacements of the member.

Physically these coefficients represent the load on the member when the member undergoes a specified unit displacement.

For example, if $d_{Ny'} = 1$, *while all other displacements are zero,* the member will be subjected only to the four loadings indicated in the first column of the **k** matrix.

In a similar manner, the other columns of the **k** matrix are the member loadings for unit displacements identified by the degree-of-freedom code numbers listed above the columns. From the development, both equilibrium and compatibility of displacements have been satisfied.

Also, it should be noted that this matrix is the ***same*** in both the **local** and **global coordinates** since the x', y', z' axes are parallel to x, y, z and, therefore, transformation matrices are not needed between the coordinates.

Beam-Structure Stiffness Matrix

Once all the member stiffness matrices have been found, assemble them into the structure stiffness matrix **K.**

This process depends on first knowing the ***location*** of each element in the member stiffness matrix.

The rows and columns of each **k** matrix are identified by the two code numbers at the near end of the member $(N_{y'}, N_z)$ followed by those at the far end $(F_{y'}, F_z)$. Therefore, when assembling the matrices, each element must be placed in the same location of the **K** matrix.

In this way, **K** will have an order that will be equal to the highest code number assigned to the beam, since this represents the total number of degrees of freedom.

Where several members are connected to a node, their member stiffness influence coefficients will have the same position in the K matrix and therefore must be algebraically added together to determine the nodal stiffness influence coefficient for the structure.

Application of the Stiffness Method for Beam Analysis

After the structure stiffness matrix is determined, the loads at the nodes of the beam can be related to the displacements using the structure stiffness equation

$$\mathbf{Q} = \mathbf{K}\mathbf{D}$$

Here **Q** and **D** are column matrices that represent both the known and unknown loads and displacements. Partitioning the stiffness matrix into the known and unknown elements of load and displacement, we have

$$\left[\begin{array}{c} \mathbf{Q}_k \\ \hline \mathbf{Q}_u \end{array}\right] = \left[\begin{array}{c|c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hline \mathbf{K}_{21} & \mathbf{K}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{D}_u \\ \hline \mathbf{D}_k \end{array}\right]$$

Which when expanded yields the two equations

$$\boxed{Q_k = K_{11} D_u + K_{12} D_k \qquad Q_u = K_{21} D_u + K_{22} D_k}$$

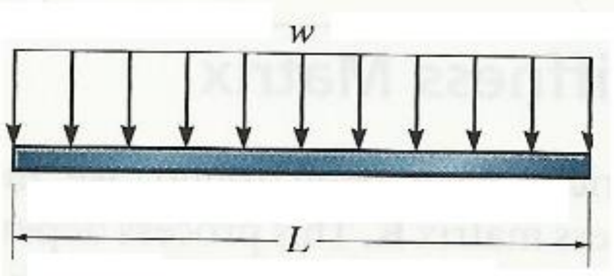
The unknown displacements Du are determined from the first of these equations. Using these values, the support reactions Qu are computed for the second equation.

Intermediate Loadings

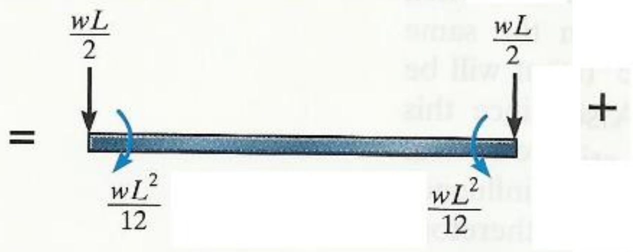
For application, it is important that the elements of the beam be free of loading along its length. This is necessary since the stiffness matrix for each element was developed for loadings applied only at its ends. Oftentimes, however, beams will support a distributed loading, and this condition will require modification in order to perform the matrix analysis.

To handle this case, we will use the principle of superposition in a manner similar to that used for trusses.

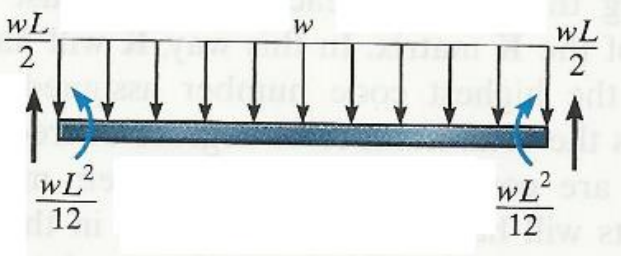
To show its application, consider the beam element of length L in Fig., which is subjected to the uniform distributed load w. First we will apply fixed-end moments and reactions to the element, which will be used in the stiffness method. We will refer to these loadings as a column matrix — q0. Then the distributed loading and its reactions q0 are applied. The actual loading within the beam is determined by adding these two results. The fixed-end reactions for other cases of loading are given on the inside back cover.



actual loading



fixed-end element loading on joints



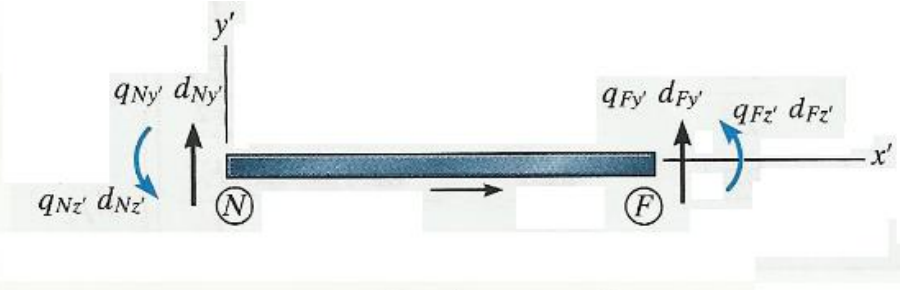
actual loading and reactions on fixed-supported element

Member Forces.

The shear and moment at the ends of each beam element can be determined using Eq. **q** = **kd** and adding on any fixed-end reactions $\mathbf{q}_0$ if the element is subjected to an intermediate loading. We have

$$\mathbf{q} = \mathbf{k}\mathbf{d} + \mathbf{q}_0$$

If the results are negative, it indicates that the loading acts in the opposite direction to that shown



positive sign convention

PROCEDURE FOR ANALYSIS

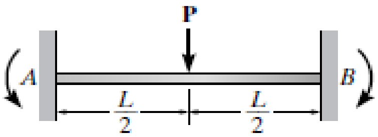
- **Divide** the beam into finite **elements** and arbitrarily identify each element and its **nodes**. Use a number written in a circle for a node and a number written in a square for a member. Usually an element extends between points of support, points of concentrated loads, and joints, or to points where internal loadings or displacements are to be determined. Also, E and I for the elements must be constants.
- Specify the **near** and **far ends** of each element symbolically by directing an **arrow** along the element, with the head directed toward the far end.
- At each **nodal** point specify numerically the **y** and **z** code numbers. In all cases use the ***lowest code numbers*** to identify all the **unconstrained degrees of freedom**, followed by the remaining or **highest numbers** to identify the degrees of freedom that are **constrained**.
- From the problem, establish the **known displacements** $\boldsymbol{D_k}$ and **known external loads** $\boldsymbol{Q_k}$. Include any *reversed* fixed-end loadings if an element supports an intermediate load.
- Determine the **stiffness matrix** for each **element** expressed in global coordinates.

$$\begin{array}{c}
\begin{array}{cccc} y & z & y & z \end{array} \\
\begin{bmatrix}
\frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\end{array}$$

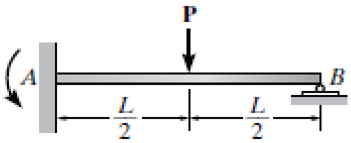
- After each member stiffness matrix is determined, and the rows and columns are identified with the appropriate code numbers, assemble the matrices to determine the **structure stiffness matrix K.**
- As a partial check, the member *and* structure stiffness matrices should all be *symmetric.*
- **Partition** the structure stiffness equation and carry out the matrix multiplication in order to determine the **unknown displacements** $\mathbf{D_u}$ and support **reactions** $\mathbf{Q_u}$.
- The internal shear and moment **q** at the ends of each beam element can be determined, accounting for the additional fixed-end loadings.

$$\mathbf{q = kd + q_0}$$

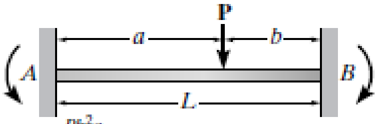
Fixed end moments



$(FEM)_{AB} = \frac{PL}{8}$ $(FEM)_{BA} = \frac{PL}{8}$

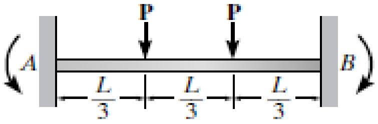


$(FEM)'_{AB} = \frac{3PL}{16}$

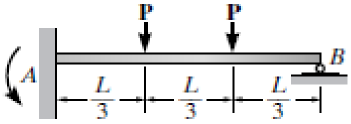


$(FEM)_{AB} = \frac{Pb^2a}{L^2}$ $(FEM)_{BA} = \frac{Pa^2b}{L^2}$

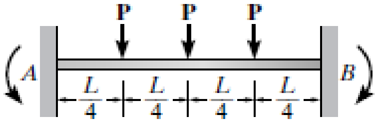
$(FEM)'_{AB} = \left(\frac{P}{L^2}\right)\left(b^2a + \frac{a^2b}{2}\right)$



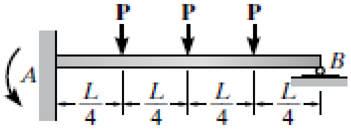
$(FEM)_{AB} = \frac{2PL}{9}$ $(FEM)_{BA} = \frac{2PL}{9}$



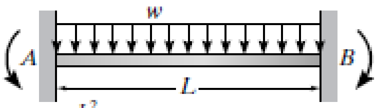
$(FEM)'_{AB} = \frac{PL}{3}$



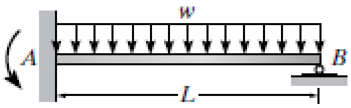
$(FEM)_{AB} = \frac{5PL}{16}$ $(FEM)_{BA} = \frac{5PL}{16}$



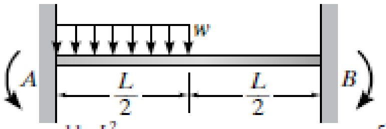
$(FEM)'_{AB} = \frac{15PL}{32}$



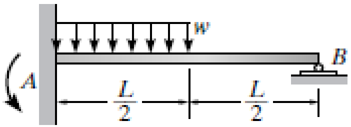
$(FEM)_{AB} = \frac{wL^2}{12}$ $(FEM)_{BA} = \frac{wL^2}{12}$



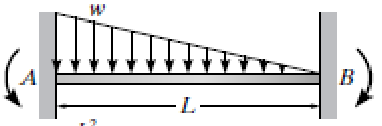
$(FEM)'_{AB} = \frac{wL^2}{8}$



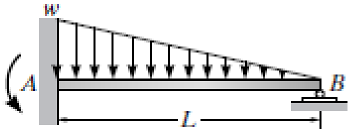
$(FEM)_{AB} = \frac{11wL^2}{192}$ $(FEM)_{BA} = \frac{5wL^2}{192}$



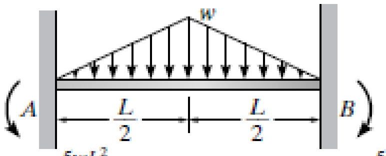
$(FEM)'_{AB} = \frac{9wL^2}{128}$



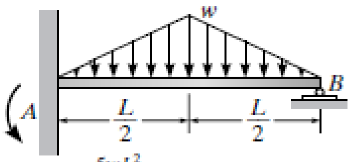
$(FEM)_{AB} = \frac{wL^2}{20}$ $(FEM)_{BA} = \frac{wL^2}{30}$



$(FEM)'_{AB} = \frac{wL^2}{15}$



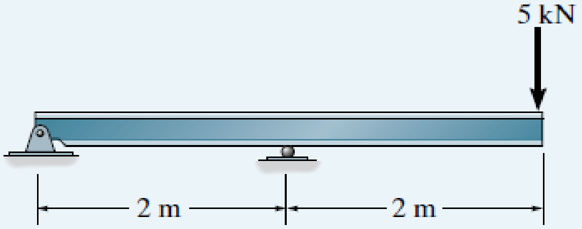
$(FEM)_{AB} = \frac{5wL^2}{96}$ $(FEM)_{BA} = \frac{5wL^2}{96}$



$(FEM)'_{AB} = \frac{5wL^2}{64}$

Example

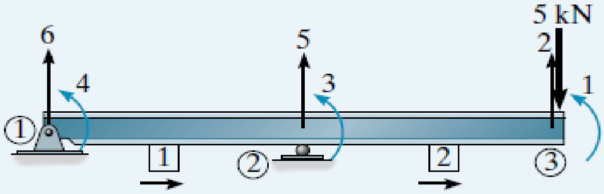
Determine the reactions at the supports of the beam shown in Fig. 15–8*a*. *EI* is constant.



Notation. The beam has two elements and three nodes, which are identified in Fig. 15–8*b*. The code numbers 1 through 6 are indicated such that the *lowest numbers* 1–4 *identify the unconstrained degrees of freedom.*

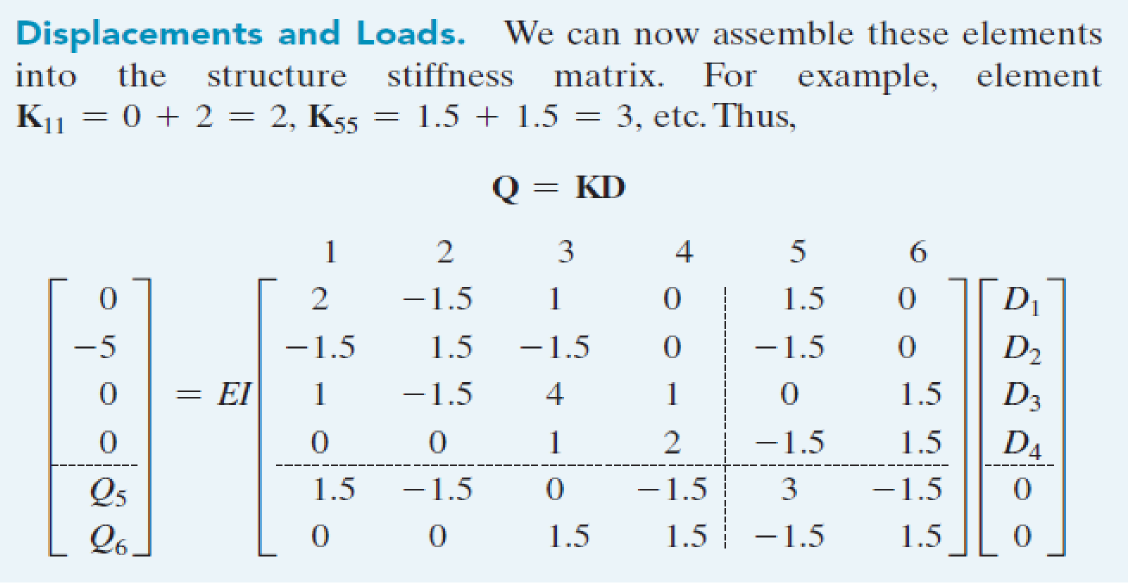
The known load and displacement matrices are

$$\mathbf{Q}_k = \begin{array}{c} \left[\begin{array}{c} 0 \\ -5 \\ 0 \\ 0 \end{array}\right] \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \mathbf{D}_k = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\begin{array}{c} 5 \\ 6 \end{array}$$



Member Stiffness Matrices. Each of the two member stiffness matrices is determined from Eq. 15–1. Note carefully how the code numbers for each column and row are established.

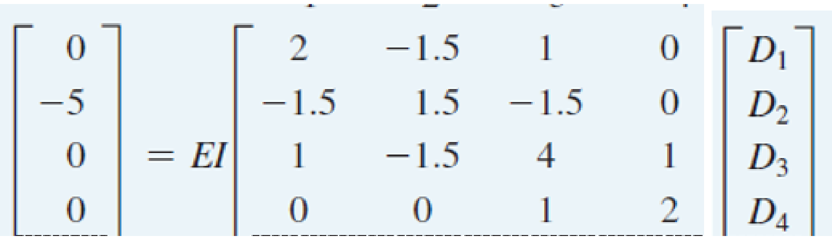
$$\mathbf{k}_1 = EI\begin{array}{c} \begin{array}{cccc} 6 & 4 & 5 & 3 \end{array} \\ \left[\begin{array}{cccc} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{array}\right] \end{array}\begin{array}{c} \\ 6 \\ 4 \\ 5 \\ 3 \end{array} \qquad \mathbf{k}_2 = EI\begin{array}{c} \begin{array}{cccc} 5 & 3 & 2 & 1 \end{array} \\ \left[\begin{array}{cccc} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{array}\right] \end{array}\begin{array}{c} \\ 5 \\ 3 \\ 2 \\ 1 \end{array}$$



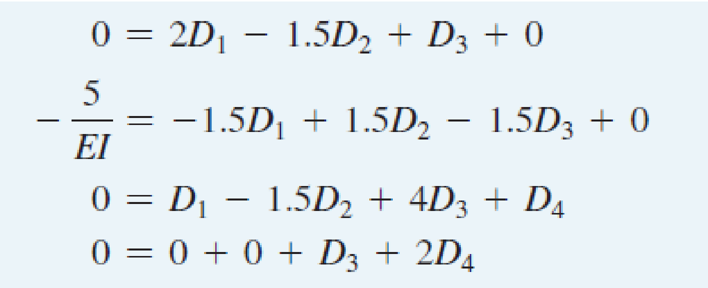
Displacements and Loads. We can now assemble these elements into the structure stiffness matrix. For example, element $\mathbf{K}_{11} = 0 + 2 = 2$, $\mathbf{K}_{55} = 1.5 + 1.5 = 3$, etc. Thus,

$$\mathbf{Q} = \mathbf{KD}$$

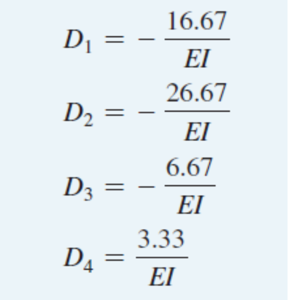
$$\left[\begin{array}{c} 0 \\ -5 \\ 0 \\ 0 \\ \hline Q_5 \\ Q_6 \end{array}\right] = EI \begin{array}{cccc:cc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array} \left[\begin{array}{cccc:cc} 2 & -1.5 & 1 & 0 & 1.5 & 0 \\ -1.5 & 1.5 & -1.5 & 0 & -1.5 & 0 \\ 1 & -1.5 & 4 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 2 & -1.5 & 1.5 \\ \hdashline 1.5 & -1.5 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 1.5 & 1.5 & -1.5 & 1.5 \end{array}\right] \left[\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ \hline 0 \\ 0 \end{array}\right]$$



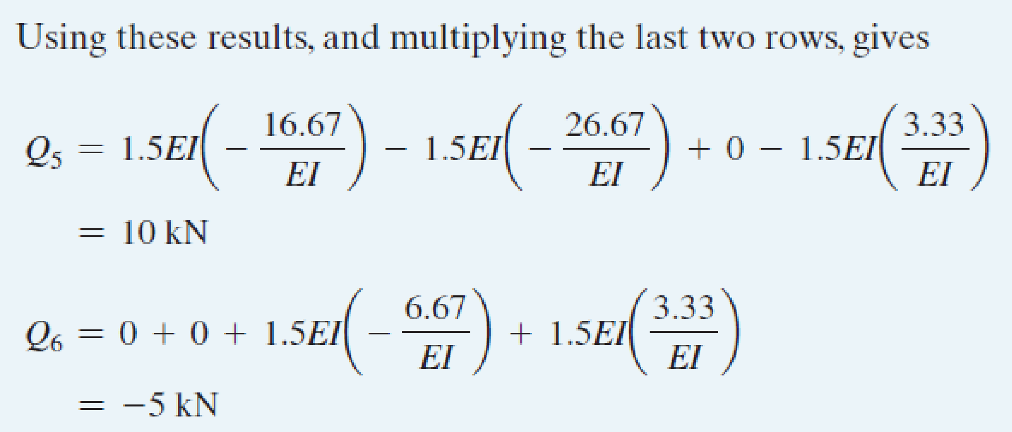
$$\begin{bmatrix} 0 \\ -5 \\ 0 \\ 0 \end{bmatrix} = EI \begin{bmatrix} 2 & -1.5 & 1 & 0 \\ -1.5 & 1.5 & -1.5 & 0 \\ 1 & -1.5 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix}$$



$$0 = 2D_1 - 1.5D_2 + D_3 + 0$$
$$-\frac{5}{EI} = -1.5D_1 + 1.5D_2 - 1.5D_3 + 0$$
$$0 = D_1 - 1.5D_2 + 4D_3 + D_4$$
$$0 = 0 + 0 + D_3 + 2D_4$$



$$D_1 = -\frac{16.67}{EI}$$
$$D_2 = -\frac{26.67}{EI}$$
$$D_3 = -\frac{6.67}{EI}$$
$$D_4 = \frac{3.33}{EI}$$



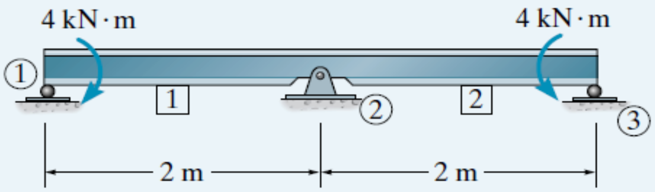
Using these results, and multiplying the last two rows, gives

$$Q_5 = 1.5EI\left(-\frac{16.67}{EI}\right) - 1.5EI\left(-\frac{26.67}{EI}\right) + 0 - 1.5EI\left(\frac{3.33}{EI}\right)$$
$$= 10 \text{ kN}$$

$$Q_6 = 0 + 0 + 1.5EI\left(-\frac{6.67}{EI}\right) + 1.5EI\left(\frac{3.33}{EI}\right)$$
$$= -5 \text{ kN}$$

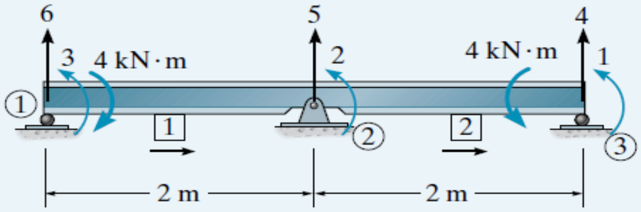
Example

The beam in Fig. 15–10*a* is subjected to the two couple moments. If the center support ② settles 1.5 mm, determine the reactions at the supports. Assume the roller supports at ① and ③ can pull down or push up on the beam. Take $E = 200$ GPa and $I = 22(10^{-6})\ \text{m}^4$.



Notation. The beam has two elements and three unknown degrees of freedom. These are labeled with the lowest code numbers, Fig. 15–10*b*. Here the known load and displacement matrices are

$$\mathbf{Q}_k = \begin{array}{c} \left[\begin{array}{r} 4 \\ 0 \\ -4 \end{array}\right] \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad \mathbf{D}_k = \left[\begin{array}{c} 0 \\ -0.0015 \\ 0 \end{array}\right]\begin{array}{c} 4 \\ 5 \\ 6 \end{array}$$



$$\mathbf{k}_1 = EI \begin{array}{c} \begin{array}{cccc} 6 & 3 & 5 & 2 \end{array} \\ \left[\begin{array}{rrrr} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 3 \\ 5 \\ 2 \end{array}$$

$$\mathbf{k}_2 = EI \begin{array}{c} \begin{array}{cccc} 5 & 2 & 4 & 1 \end{array} \\ \left[\begin{array}{rrrr} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 2 \\ 4 \\ 1 \end{array}$$

$$\left[\begin{array}{c} 4 \\ 0 \\ -4 \\ \hdashline Q_4 \\ Q_5 \\ Q_6 \end{array}\right] = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{rrr:rrr} 2 & 1 & 0 & -1.5 & 1.5 & 0 \\ 1 & 4 & 1 & -1.5 & 0 & 1.5 \\ 0 & 1 & 2 & 0 & -1.5 & 1.5 \\ \hdashline -1.5 & -1.5 & 0 & 1.5 & -1.5 & 0 \\ 1.5 & 0 & -1.5 & -1.5 & 3 & -1.5 \\ 0 & 1.5 & 1.5 & 0 & -1.5 & 1.5 \end{array}\right] \end{array} \left[\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ \hdashline 0 \\ -0.0015 \\ 0 \end{array}\right]$$

Solving for the unknown displacements,

$$\frac{4}{EI} = 2D_1 + D_2 + 0D_3 - 1.5(0) + 1.5(-0.0015) + 0$$

$$0 = 1D_1 + 4D_2 + 1D_3 - 1.5(0) + 0 + 0$$

$$\frac{-4}{EI} = 0D_1 + 1D_2 + 2D_3 + 0 - 1.5(-0.0015) + 0$$

Substituting $EI = 200(10^6)(22)(10^{-6})$, and solving,

$$D_1 = 0.001580 \text{ rad}, \quad D_2 = 0, \quad D_3 = -0.001580 \text{ rad}$$

Using these results, the support reactions are therefore

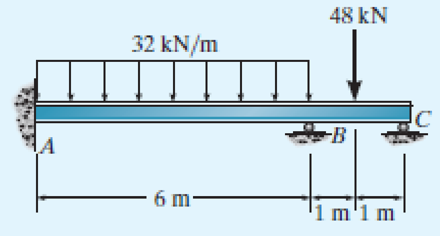
$$Q_4 = 200(10^6)22(10^{-6})[-1.5(0.001580) - 1.5(0) + 0 + 1.5(0) - 1.5(-0.0015) + 0] = -0.525 \text{ kN}$$

$$Q_5 = 200(10^6)22(10^{-6})[1.5(0.001580) + 0 - 1.5(-0.001580) - 1.5(0) + 3(-0.0015) - 1.5(0)] = 1.05 \text{ kN}$$

$$Q_6 = 200(10^6)22(10^{-6})[0 + 1.5(0) + 1.5(-0.001580) + 0 - 1.5(-0.0015) + 1.5(0)] = -0.525 \text{ kN}$$

Example

Determine the moment developed at support A of the beam shown in Fig. 15–11a. Assume the roller supports can pull down or push up on the beam. Take $E = 200$ GPa, $I = 216(10^6)$ mm^4.

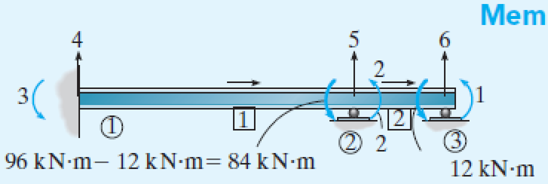


SOLUTION

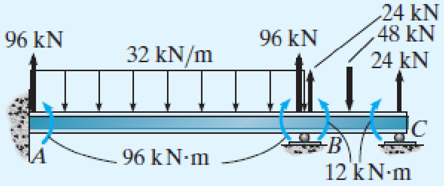
Notation. Here the beam has two unconstrained degrees of freedom, identified by the code numbers 1 and 2.

The matrix analysis requires that the external loading be applied at the nodes, and therefore the distributed and concentrated loads are replaced by their equivalent fixed-end moments, which are determined from the table on the inside back cover. (Here $wL^2/12 = 96$ kN · m and $PL/8 = 12$ kN · m.) Note that no external loads are placed at node ① and no external vertical forces are placed at nodes ② and ③, since the reactions at code numbers 3 through 6 *are to be unknowns* in the load matrix. Using superposition, the results of the matrix analysis for the loading in Fig. 15–11b will later be modified by the fixed-end loads shown in Fig. 15–11c. From Fig. 15–11b, the known displacement and load matrices are

$$\mathbf{D}_k = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \qquad \mathbf{Q}_k = \begin{bmatrix} 12 \\ 84 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$



beam to be analyzed by stiffness method
(b)



beam subjected to actual load and
fixed-supported reactions
(c)

Fig. 15–11

Member Stiffness Matrices. Each of the two member stiffness matrices is determined from Eq. 15–1.

Member 1.

$$\frac{12EI}{L^3} = \frac{[200(10^6)][216(10^{-6})]}{6^3} = 2400$$

$$\frac{6EI}{L^2} = \frac{[200(10^6)][216(10^{-6})]}{6^2} = 72000$$

$$\frac{4EI}{L} = \frac{[200(10^6)][216(10^{-6})]}{6} = 28\,800$$

$$\frac{2EI}{L} = \frac{[200(10^6)][216(10^{-6})]}{6} = 14\,400$$

$$\mathbf{k}_1 = \begin{array}{cc} \begin{array}{cccc} 4 & 3 & 5 & 2 \end{array} & \\ \left[\begin{array}{cccc} 2400 & 7200 & -2400 & 7200 \\ 7200 & 28\,800 & -7200 & 14\,400 \\ -2400 & -7200 & 2400 & -7200 \\ 7200 & 14\,400 & -7200 & 28\,800 \end{array}\right] & \begin{array}{c} 4 \\ 3 \\ 5 \\ 2 \end{array} \end{array}$$

Member 2.

$$\frac{12EI}{L^3} = \frac{[200(10^6)][216(10^{-6})]}{2^3} = 64\,800$$

$$\frac{6EI}{L^2} = \frac{[200(10^6)][216(10^{-6})]}{2^2} = 64\,800$$

$$\frac{4EI}{L} = \frac{4[200(10^9)][216(10^{-6})]}{2} = 86\,400$$

$$\frac{2EI}{L} = \frac{2[200(10^9)][216(10^{-6})]}{2} = 43\,200$$

$$\mathbf{k}_2 = \begin{array}{cc} \begin{array}{cccc} 5 & 2 & 6 & 1 \end{array} & \\ \left[\begin{array}{cccc} 64\,800 & 64\,800 & -64\,800 & 64\,800 \\ 64\,800 & 86\,400 & -64\,800 & 43\,200 \\ -64\,800 & -64\,800 & 64\,800 & -64\,800 \\ 64\,800 & 43\,200 & -64\,800 & 86\,400 \end{array}\right] & \begin{array}{c} 5 \\ 2 \\ 6 \\ 1 \end{array} \end{array}$$

Displacements and Loads. We require

$$\mathbf{Q} = \mathbf{KD}$$

$$\left[\begin{array}{c} 12 \\ 84 \\ \hline Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{array}\right] = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cc|cccc} 86\,400 & 43\,200 & 0 & 0 & 64\,800 & -64\,800 \\ 43\,200 & 115\,200 & 14\,400 & 7200 & 57\,600 & -64\,800 \\ \hline 0 & 14\,400 & 28\,800 & 7200 & -7200 & 0 \\ 0 & 7200 & 7200 & 7200 & -2400 & 0 \\ 64\,800 & 57\,600 & -7200 & -2400 & 67\,200 & -64\,800 \\ -64\,800 & -64\,800 & 0 & 0 & -64\,800 & 64\,800 \end{array}\right] \end{array} \left[\begin{array}{c} D_1 \\ D_2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Solving in the usual manner,

$$
\begin{aligned}
12 &= 86\,400D_1 + 43\,200D_2 \\
84 &= 43\,200D_1 + 115\,220D_2 \\
D_1 &= -0.2778(10^{-3})\text{ m} \\
D_2 &= 0.8333(10^{-3})\text{ m}
\end{aligned}
$$

Thus,

$$Q_3 = 0 + 14\,400(0.8333)(10^{-3}) = 12\text{ kN}\cdot\text{m}$$

The actual moment at A must include the fixed-supported *reaction* of $+96$ kN · m shown in Fig. 15–11*c*, along with the calculated result for Q_3. Thus,

$$M_{AB} = 12\text{ kN}\cdot\text{m} + 96\text{ kN}\cdot\text{m} = 108\text{ kN}\cdot\text{m}\circlearrowleft \qquad \textit{Ans.}$$

This result compares with that determined in Example 10.2.

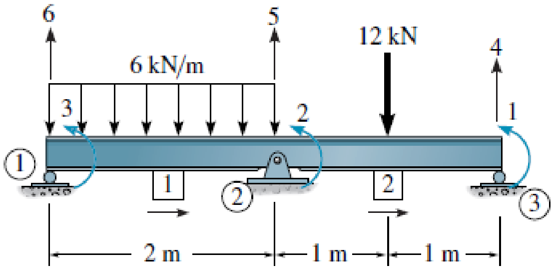
Although not required here, we can determine the internal moment and shear at B by considering, for example, member 1, node 2, Fig. 15–11*b*. The result requires expanding

$$\mathbf{q}_1 = \mathbf{k}_1\mathbf{d} + (\mathbf{q}_0)_1$$

$$
\begin{bmatrix} q_4 \\ q_3 \\ q_5 \\ q_2 \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 4 & 3 & 5 & 2 \end{array} \\
\begin{bmatrix}
2400 & 7200 & -2400 & 7200 \\
7200 & 28\,800 & -7200 & 14\,400 \\
-2400 & -7200 & 2400 & -7200 \\
7200 & 14\,400 & -7200 & 28\,800
\end{bmatrix}
\end{array}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.8333 \end{bmatrix}
(10^{-3}) +
\begin{bmatrix} 96 \\ 96 \\ 96 \\ -96 \end{bmatrix}
$$

HW:

- Determine the reactions at the supports. *EI* is constant.



- Determine the moments at the supports. Assume ② is a roller. EI is constant.