زانكۆی سەلاحەدین – هەولێر

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The Fibonacci Numbers

Research Project

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Supervisor's Certification

I certify that this thesis was prepared under my supervision at department of mathematics, College of Education in Salahaddin University – Erbil, and that, in my opinion it is fully adequate in scope and quality as a thesis for the degree of bachelor in mathematics.

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Content

Abstract

In this report we study the Fibonacci numbers and it will give some properties about it. The Fibonacci sequence is a sequence in which each term is the sum of the two numbers preceding it.

Introduction

Perhaps the greatest mathematician of the Middle Ages was Leonardo of Pisa (1180– 1250), who wrote under the name of Fibonacci—a contraction of “filiusBonacci,” that is, Bonacci’s son. Fibonacci was born in Pisa and educated in North Africa, where his father was in charge of a customhouse. In the expectation of entering the mercantile business, the youth traveled about the Mediterranean visiting Spain, Egypt, Syria, and Greece. The famous Liber Abaci, composed upon his return to Italy, introduced the Latin West to Islamic arithmetic and algebraic mathematical practices. A briefer work of Fibonacci’s, the Liber Quadratorum (1225), is devoted entirely to Diophantine problems of second degree. It is regarded as the most important contribution to Latin Middle-Ages number theory before the works of Bache and Fermat. Like those before him, Fibonacci allows (positive) real numbers as solutions. One problem, for instance, calls for finding a square that remains square when increased or decreased by 5; that is, obtain a simultaneous solution to the pair of equations $x^2 + 5 = y^2$, $x^2 - 5 = z^2$, where x, y, z are unknowns. Fibonacci gave 41/12 as an answer, for $(41/12)^2 + 5 = (49/12)^2$, $(41/12)^2 - 5 = (31/12)^2$

Chapter one

Background

In this chapter we take some known definitions and results that are we needed

Definition1.1[1]: (Division Algorithm). Given integers a and b, with $b > 0$, there exist unique integers q and r satisfying

$$a = bq + r \qquad\qquad 0 \le r < b.$$

Definition1.2 [1]: Let a and b be given integers, with at lest one of them different zero. The greatest common divisor of a and b denoted by $gcd\ (a, b)$ is the positive integer d satisfying

1. $d|a$ and $d|b$
2. If $c|a$ and $c|b$ then $c \le d$

Example1.3:

The positive divisors of 8 are $1,2,4\ and$ 8.

Corollary 1.4 [1]: if $gcd(a, b) = d$, then gcd $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

Corollary 1.5[1]: If $a \mid c$ and $b \mid c$, then $ab \mid c$.

Theorem 1.6[1]: If $a \mid bc$, with $\gcd(a, b) = 1$, then $a \mid c$.

Lemma1.8 [1]: If $a = bq + r$, then $gcd(a, b) = gcd(b, r)$.

Example1.9: $gcd(12378{,}3054) = 6$.

$$12378 = 4.3045 + 162$$

$$3054 = 18.162 + 138$$

$$162 = 1.138 + 24$$

$$138 = 5.24 + 18$$

$$24 = 1.18 + 6$$

$$18 = 3.6 + 0$$

The construction of Fibonacci sequence1.9 [1]:

A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive?

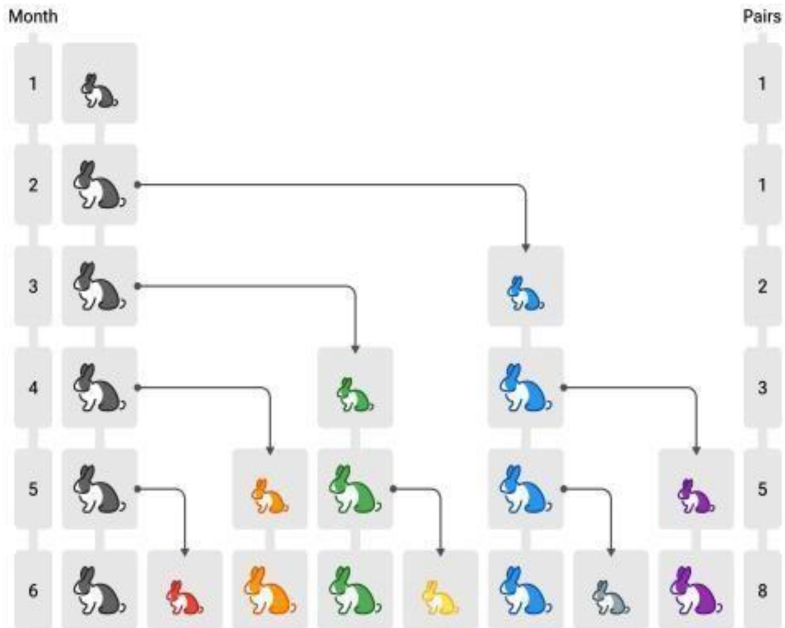
Assuming that none of the rabbits dies, then a pair is born during the first month, so that there are two pairs present. During the second month, the original pair has produced another pair. One month later, both the original pair and the firstborn pair have produced new pairs, so that three adult and two young pairs are present, and so on. (figure1) The point to bear in mind is that each month the young pairs grow up and become adult pairs, making the new “adult” entry the previous one plus the previous “young” entry. Each of the pairs that was adult last month produces one young pair, so that the new “young” entry is equal to the previous “adult” entry when continued indefinitely, the sequence encountered in the rabbit problem

1,1,2,3,5,8,13,21,34,55,89,144,233,377...

is called the Fibonacci sequence and its terms the Fibonacci numbers.

Growth of rabbit colony

Months.	Adult.	Young pairs.	Total
1	1	1	2
2	2	2	3
3	3	2	5
4	5	3	8
5	8	5	13
6	13	8	21
7	21	13	34
8	34	21	55
9	55	34	89
10	89	55	144
11	144	89	233
12	233	144	377


Month
1
2
3
4
5
6
Pairs
1
1
2
3
5
8

Chapter Two

Section One

Simple Properties of the Fibonacci Numbers

Definition 2.1.1 [3]: A Fibonacci number is a positive number of the form

$$u_n = u_{n-1} + u_{n-2} \text{ for } n > 2.$$

Remark2.1.2 [1]: The Fibonacci sequence exhibits an intriguing property, namely,

$$2 = 1 + 1 \; or \;\; u_3 = u_2 + u_1$$

$$3 = 2 + 1 \; or \;\; u_4 = u_3 + u_2$$

$$5 = 3 + 2 \; or \;\; u_5 = u_4 + u_3$$

$$8 = 5 + 3 \; or \;\; u_6 = u_5 + u_4$$

By this time, the general rule of formulation should be discernible:

$u_1 = u_2 = 1$ and $u_n = u_n - 1 + u_n - 2 \quad for \; n \geq 3$.

Theorem 2.1.3 [1]: For the Fibonacci sequence, $gcd\ (u_n, u_n+1) = 1$ for every $n \geq 1$.

Proof: Let us suppose that the integer $d > 1$ divides both u_n and u_n+1. Then their difference $u_n+1 - u_n = u_n-1$ is also divisible by d. From this and from the relation $un - un - 1 = un - 2$, it may be concluded that $d \mid u_n-2$**.** Working backward, the same argument shows that $d \mid u_n-3$, $d \mid u_n-4$ ... and finally that d $|u_1$. But $u_1 = 1$ **,** which is certainly not divisible by any $d > 1$. This contradiction ends our proof.

Example 2.1.4: in the Fibonacci sequence we take $u_5 = 5,\ u_6 = 8,\ u_7 = 13$ we see that $gcd\ (u_5, u_6) = 1$. and $gcd(\ u_6, u_7\ ) = 1$.

Theorem 2.1.5 [1]: For $n \geq 1$, the Fibonacci number $u_{5n+2} > 10^n$.

Proof: We have $u_7 > 10$, $u_{12} > 100$, $u_{17} > 1000$, $u_{22} > 10000$...

Inequality can be established using induction on n, the case $n = 1$ being obvious because $u_7 = 13 > 10$. Now assume that the inequality holds for an arbitrary integer n; we wish to show that it also holds for n + 1. The recursion rule $u_k = u_{k}$-1 + u_k-2 can be used several times to express u_5(n+1) +2 $= u_{5n}$+7 in terms of previous Fibonacci numbers to arrive at

$$u_{5n+7} = 8u_{5n+2} + 5u_{5n+1}$$

$$> 8u_{5n+2} + 2(u_{5n+1} + u_{5n})$$

$$= 10u_{5n+2}$$

$$> 10 \cdot 10^n = 10^{n+1}.$$

Hence $u_{5n+2} > 10^n$

Lemma2.1.6 [4]: Another important formula

$$u_n + u_m = u_{n-1}u_m u_n u_{m+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Proof: we will now begin this proof by induction on m. for m=1

$$u_{n+1} = u_{n-1} - u_n$$

$$= u_{-n-1}u_1 + u_n u_2,$$

Which we can holds true to the formula. The equation for $m = 2$ also proves True for our formula, as

$$u_{n+2} = u_{n+1} + u_n,$$

$$= u_{n-1} + u_n + u_n$$

$$= u_{n-1} + 2u_n$$

$$= u_{n-1}u_2 + u_n u_3$$

Now suppose our formula to be true for $m = k$ and for $m = K + 1$.

So by induction, assume

$$u_{n+2} = u_{n-1}u_k + u_n u_{n+1}$$

And

$$u_{n+k}+1 = u_{n-1}u_{k+1} + u_n u_{k+2}.$$

If we add these two equations terms by term, we obtain

$$u_{n+k} + u_{n+k+1} = u_{n-1}(u_k + u_{k+1}) + u_n(u_{k+1} + u_{k+2})$$

$$u_n + u_{k+2} = u_{n-1}(u_{k+2} + u_n u_{k+3},$$

So, by induction we have proven our initial formula

Example 2.1.7: $u_9 = u_{6+3} = u_5 u_3 + u_6 u_4 = 5.2 + 8.3 = 10 + 24 = 34.$

Theorem 2.1.8 [1]: For $m \geq 1, n \geq 1, u_{mn}$ is divisible by u_m.

Proof: We again argue by induction on n, the result being certainly true when $n = 1$. For our induction hypothesis, let us assume that u_{mn} is divisible by u_m for $n = 1, 2 \ldots k$. The transition to the case $u_{m(k+1)} = u_{mk+m}$ is realized using Eq. (1); indeed, $u_{m(k+1)} = u_{mk-1}u_m + u_{mk}\, u_{m+1}$

Because um divides umk by supposition, the right-hand side of this expression (and, hence, the left-hand side) must be divisible by um. accordingly,

$u_m \mid u_m(k + 1)$, which was to be proved.

Example2.1.9: $u_3 = 2$ $u_{12} = 144$. Since $2 \mid 144$ $then, u_{12} \div u_3$

Lemma2.1.10 [1]: If $m = qn + r$, then $gcd(u_m, u_n) = gcd(u_r, u_n)$.

Proof: To begin with, Eq. (1) allows us to write

$$gcd(u_m, u_n) = gcd(u_{qn+r}, u_n)$$

$$= gcd(u_{qn-1}u_r + u_{qn}\, u_{r+1}, u_n)$$

An appeal to Theorem 1.2 and the fact that $gcd(a + c, b) = gcd(a, b)$, whenever $b \mid c$, gives

$$gcd(u_{qn-1}u_r + u_{qn}u_{r+1}, u_n) = gcd(u_{qn-1}u_r, u_n)$$

Our claim is that $gcd(u_{qn-1}, u_n) = 1$. To see this, set $d = gcd(u_{qn-1}, u_n)$. the relations $d|u_n$ and $u_n|u_{qn}$ imply that $d|uqn$, and therefore d is a (positive) common divisor of the successive Fibonacci numbers u_{qn-1} and u_{qn}. Because successive Fibonacci numbers are relatively prime, the effect of this is that d = 1. To finish the proof, the reader is left the task of showing that when $gcd(a, c) = 1$, then $gcd(a, bc) = gcd(a, b)$. Knowing this, we can immediately pass on to $gcd(u_m, u_n) = gcd(u_{qn-1}u_r, u_n) = gcd(u_r, u_n)$ the desired equality.

Theorem 2.1.11 [1]: The greatest common divisor of two Fibonacci numbers is again a Fibonacci number; specifically, $gcd(u_m, u_n) = u_d$ where $d = gcd(m, n)$

Proof: Assume that$m \geq n$. Applying the Euclidean Algorithm to m and n, we get the following system of equations:

$$m = q_1 n + r_1 \qquad 0 < r_1 < n$$

$$n = q_2 r_1 + r2. \qquad 0 < r_2 < r_1$$

$$r_1 = q_3 r_2 + r_3 \qquad 0 < r_3 < r_2$$

$gcd(u_m, u_n) = gcd(u_{r1}, u_n) = gcd(u_{r1}, u_{r2}) = \cdots = gcd(u_{rn}-1, u_{rn})$

Because $r_n|r_{n-1}$, Theorem1.2 tell s us that $u_{r_n} \Big| u_{r_{n-1}}$, whence $gcd(u_{r_n}, u_{r_{n-1}}) = u_{rn}$. But r_n, being the last nonzero remainder in the Euclidean Algorithm for m and n, is equal to $gcd(m, n)$. Tying up the loose ends, we get $gcd(u_m, u_n) = u\, gcd(m, n)$ and in this way the theorem is established.

Example2.1.12: A good illustration of Theorem 1.3 is provided by calculating $gcd(u_{16}, u_{12}) = gcd\ (987, 144)$. From the Euclidean Algorithm,

$$987 = 6\cdot 144+123$$

$$144 = 1\cdot 123+21$$

$$123 = 5\cdot 21+18$$

$$21 = 1\cdot 18+3$$

$$18 = 6 \cdot 3 + 0$$

And there fore $gcd(987, 144) = 3$. The net result is that

$$gcd(u_{16}, u_{12}) = 3 = u_4 = u\ gcd(16,12).$$

Corollary.2.1.13 [1]: In the Fibonacci sequence, $u_m|u_n$ if and only if $m|n$ for $n \geq m \geq 3$.

Section Two: Sum of Fibonacci Numbers

Lemma2.2.1 [4]: Sum of the Fibonacci numbers

$$u_1 + u_2 + u_3 + \ldots + u_n = u_{n+2} - 1.$$

Proof: From the definition of the Fibonacci sequence, we know

$u_n+1 = u_n + u_{n-1}, n \geq 1, that\ is\ u_{n-1} = u_{n-1} - u_n$

$$u_1 = u_3 - u_2$$

$$u_2 = u_4 - u_3$$

$$u_3 = u_5 - u_4$$

.

.

.

$$u_n - 1 = u_{n+1} - u_n$$

$$u_n = u_{n+2} - u_{n+1}$$

We know add these equations to find

$$u_1 + u_2 + \ldots + u_{n-1} + u_n = u_{n+2} - 1$$

If $u_1 = 1$, this equations to equivalent to our initial conjecture of

$$u_1 + u_2 + \ldots + u_{n-1} + un = u_{n+2} - 1$$

Lemma2.2.2 [4]: Sum of the odd terms

The sum of the odd terms of the Fibonacci sequence

$$u_1 + u_3 + u_5 + \ldots + u_{2n-1} = u_{2n}$$

Proof: From definition of the sequence

$$u_1 = u_2$$

$$u_3 = u_4 - u_2$$

$$u_5 = u_6 - u_4$$

.

.

.

$$u_{2n} - 1 = u_{2n} - u_{2n} - 2$$

Lemma2.2.3 [4]: sum of even terms

The sum of the even terms of the Fibonacci sequence

$$u_2 + u_4 + \ldots + u_{2n+1} - 1$$

Lemma2.2.4 [4]: Sum of Fibonacci number whit alternating sign

$$u_1 + u_3 + \ ...\ + \ (-1)^{n+1}\ u_n = 1 + (-1)^{n+1}u_{n-1}$$

Proof: From subtract our even sum equation from our odd sum equation to find

$$u_1 - u_2 + u_3 - u_4 + \ ...\ + \ u_{2n-1} + u_{2n} = -\ u_{2n} - 1\ +\ 1$$

Now adding u_{2n+1} to both sides of this equation, we obtain

$$u_1 - u_2 + u_3 + \ ... + \ u_{2n} - u_{2n} + 1\ =\ u_{2n+1}.$$

Combining equation **(1)** and **(2),** we arrive at the sum of Fibonacci number whit

Alternating signs

$$u_1 - u_2 + u_3 + \ ...\ + \ (-1)\ n + 1\ \ u_n = \ 1 + (-1)^{n+1}u_{n-1}$$

Lemma2.2.5 [4]: Difference of sequence of Fibonacci numbers

$$u_{2n} = u\,^2_{n+1} - u\,^2_{n-1}$$

Proof: let $m = n$ we obtain

$$u_{2n} = \ \ u_{n-1}\, u_n + u_n\, u_{n+1}$$

Since

$$u_n = \ u_{n+1} - u_{n-1}$$

We can now rewrite the formula as follows

Thus, we can conclude that for two Fibonacci number whose positions in the

Sequence differ by two, the difference of square will again be a Fibonacci number.

Chapter there

Music and the Fibonacci sequence [5]:

Musical scales are related to Fibonacci numbers.

Piano keyboard showing that even music is based on the Fibonacci series The Fibonacci series appears in the foundation of aspects of art, beauty and life. Even music has a foundation in the series, as: There is 13 notes in the span of any note through its octave.

A scale is composed of 8 notes, of which the 5th and 3rd notes create the basic foundation of all chords, and are based on a tone which is combination of 2 steps and 1 step from the root tone that is the 1st note of the scale.

Note too how the piano keyboard scale of C to C above of 13 keys has 8 white keys and 5 black keys, split into groups of 3 and 2.While some might "note" that there are only 12 "notes" in the scale, if you don't have a root and octave, a start and an end, you have no means of calculating the gradations in between, so this 13th note as the octave is essential to computing the frequencies of the other notes. The word "octave" comes from the Latin word for 8, referring to the eight tones of the complete musical scale, which in the key of C are C-D-E-F-G-A-B-C.

In a scale, the dominant note is the 5th note of the major scale, which is also the 8th note of all 13 notes that comprise the octave. This provides an added instance of Fibonacci numbers in key musical relationships. Interestingly, 8/13 is .61538, which approximates phi. What's more, the typical three chord song in the key of A is made up of A, its Fibonacci & phi partner E, and D, to which A bears the same relationship as E does to A. This is analogous to the "A is to B as B is to C" basis for the golden section, or in this case "D is to A as A is to E."

Here's another view of the Fibonacci relationship presented by Gerber Schwab in his YouTube video. First, number the 8 notes of the octave scale. Next, number the 13 notes of the chromatic scale. The Fibonacci numbers, in red on both scales, fall on the same keys in both methods (C, D, E, G and C). This creates the Fibonacci ratios of 1:1, 2:3, 3:5, 5:8 and 8:13:

Fibonacci piano scale 8 notes

8 notes of the octave scale

Fibonacci piano scale 13 notes

13 notes of the chromatic scale

Fibonacci piano scale ratios.

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پوختە

لەم ڕاپۆرتەدا دیراسەی ژمارەکانی فیبۆناسیمان کردووە و هەندێك لە تایبەتمەندیەکانمان وەرگرتووە. زنجیرەی فیبۆناسی زنجیرەکەیەکە هەر دانەیکی بریتیە لە کۆکراوەی دوو دانەی پێش خۆی.