زانكۆی سەلاحەدین- هەولێر

Salahaddin University-Erbil

On Multiplicative Functions

Research Project

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By

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Supervisor s Certification

I certify that this thesis was prepared under my supervision at department of mathematics, College of Education in Salahaddin University – Erbil, and that, in my opinion it is fully adequate in scope and quality as a thesis for the degree of bachelor in mathematics.

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Abstract

In this work we discuss the notion of multiplicative functions of one and two variables, Such as: Möbius and Euler's phi-function. We also take examples of such functions.

Content

List of symbols

Symbols	***Descriptions***
$a \mid b$	a divides b
$a \nmid b$	a not divides b
$gcd\ (a, b)$	greatest common divisors between a and b
Z	Set of integers
$\forall$	For all
ϕ	Euler's Phi-Function
σ	Sum of positive divisors
τ	Number of positive divisors
Π	Production
μ	Mobius common divisors
Σ	Summation

Introduction

Multiplicative arithmetic functions of a single variable are very well known in the literature. The various properties were investigated by several authors and they represent important research topics up to now.Less known are multiplicative arithmetic function of several variable of which detailed study was carried out by [R. vaidyan athas wmy] more than eighty years ago .since then many sometimes scattered results for the several variables case where published in papers and monographs and some authors of them were not aware of the paper .In fact ,there are two different notions of multiplicative function of several variables, used in last decades, both reducing to the usual multiplicativity in the one variable case. For the other concept we use the term firmly multiplicative function.

Chapter One

Background

In this section, we state some known definitions and results that we are needed in this work.

Definition1.1 [2]: Let a and b be given integers, with at lest one of them different zero. The greatest common divisor of a and b denoted by $gcd\ (a, b)$ is the positive integer d satisfying

1. $d|a$ and $d|b$
2. If $c|a$ and $c|b$ then $c \leq d$

Example1.2:

The positive divisors of 8 are $1,2,4\ and\ 8$.

Lemma 1.3[1]: Given integers a, b and c then $gcd(a, bc) = 1$ if and only if $gcd(a, b) = 1$ and $gcd(a, c)$ =1

Definition1.4 [3]: For each $a, b \in z$, we defind a congruent to b modulo n,written $a \equiv b\ (mod\ n)$, if $h\ - k$ is evenly divisible by n,so that $h - k$=ns for some $s \in z$

Example1.5: $17 \equiv 14(mod\ 3) since\ 17 - 14 = 3.1$

Corollary 1.6[1]: Any positive integer $n > 1$ can be written uniquely in a canonical from

$$n = p_1^{k1}\ p_2^{k2} \dots p_r^{kr}$$

where for i=1,2, ... r , each k_i is a positive integer and each p_i is a prime, with

$$p_1 < p_{2<} \dots < p_r$$

Theorem 1.7[1]: If n=$p_1^{k1}p_2^{k2}\ldots p_r^{kr}$ is the prime factorization of $n>1$,then the positive divisors of n are precisely those integers d of the form

$$d=p_1^{a1}p_2^{a2}\ldots p_r^{ar},$$

Where $0\leq a_i\leq k_i(i=1,2,\ldots r)$

Definition1.8 []: Given positive integer n, let $\tau(n)$ denoted the number of positive divisors of n and $\sigma(n)$ denot the sum of those divisors.

Example 1.9: consider n=15.since 15 has the positive divisor 1,3,5 and 15 then we find $\tau(15)=4\ and\ \sigma(15)=1+3+5+15=24$.

Definition 1.10.[4]: A number theoretic function f is sed to be multiplicative if $f(mn)=f(m)f(n)$ whenever $gcd\ (m,n)=1$.

Lemma1.11 []: If $gcd(m,n)\ =\ 1$, then the set of positive divisors of rnn consists of all products d_1d_2 where $d_1|n, d_2|m$,and $gcd(d_1,d_2)$= 1; furthermore, these products

are all distinct.

Example 1.12: that 3 is a primitive root of 7, for

$3^1\equiv3\quad 3^2\equiv2\quad 3^3\equiv6\quad 3^4\equiv4\quad 3^5\equiv5\quad 3^6\equiv1(\text{mod}7)$

Chapter Two

Section One: Multiplicative Functions of One Variable

In this section we study multiplicatively of some arithmetic functions and explain those by example.

Definition 2.1.1[1]: For positive integer n, define μ by the rules

$$\mu\,(n) = \begin{cases} 1, & if\ n = 1 \\ 0, & if\ \exists\ a\ prime\ p, p^2|n\ i.e\ n\ is\ not\ square\ free \\ (-1^r) & if\ n = p_1, p_2, \dots p_r\ were\ p_1, p_{2\dots}p_r\ are\ distinct\ primes \end{cases}$$

Then μ is said to be Mobius function.

Example2.1.2 [1]: Let $n = 30$ and $m = 25$.Then find $\mu(n)$ and $\mu(m)$

Solution: $\mu(30) = \mu(2.3.5) = (-1)^3 = -1$.

Since $5^2 \mid m$, then by above definition we get $\mu(25) = 0$.

Note that if p is a prime number, it is clear that $\mu(p) = -1$; also, $\mu(p^k) = 0$ for $k \geq 2$.

Theorem 2.1.4 [1]: if f is a multiplicative function F is defined by $F(n) = \sum_{d|n} f(d)$ then F is also multiplicative.

Proof: Let m and n be relative integers. Then

$$F(m,n) = \sum_{d\,|\,mn} f(d) = \sum_{\substack{d_1|m \\ d_2|n}} f(d_1 . d_2)$$

Since every divisor d of mn can be uniquely written as a product of a divisor d_1 of m and a divisor d_2 of n, where $gcd(d_1\,, d_2) \;=\; 1$. By Definition 1.10,

$$f(d_1 d_2) = f(d_1)\, f(d_2)$$

$$F(m,n) = \sum_{\substack{d_1|m \\ d_2|n}} f(d_1)f(d_2)$$

$$= (\sum_{d1|m} f(d_1))\ (\sum_{d_2} f(d_2)) = F(m)F(n)$$

Hence F is a multiplicative function

Example 2.1.5: $Let\ m = 4\ and\ n = 9,$ we have

$$F(4.9) = \sum_{d|36} f(d)$$

$$= f(1) + f(2) + f(3) + f(4) + f(6) + f(9) + f(12) + f(18) + f(36)$$

$$= f(1.1) + f(1.3) + f(1.9) + f(2.1) + f(2.3) + f(2.9)$$

$$+f(4.1) + f(4.3) + f(4.9)$$

$$= f(1)f(1) + f(1)f(3) + f(9)f(1) + f(2)f(1) + f(2)f(3) + f(2)f(9)$$

$$+f(4)f(1) + f(4)f(3) + f(4)f(9)$$

$$= [f(1) + f(2) + f(4)].[f(1) + f(3) + f(9)]$$

$$=\sum_{d|9} f(d). \sum_{d|4} f(d) = F(9)F(4)$$

Theorem 2.1.6[1]: If F is a multiplicative function and $F(n)$=$\sum_{d|n} f(d)$, then f is also multiplicative.

Theorem 2.1.7[1]: If n=$p_1^{k1}p_2^{k2} \dots p_r^{kr}$ is the prime factorization of $n > 1$ then

A) $\sigma(n)$ = ($\frac{p_1^{k1+1}}{p_1}\ \frac{p_2^{k2+1}}{p_2} \dots \frac{p_r^{kr+1}}{p_r}$)

B) $\tau(n)$ =$(k_1 + 1)(k_2 + 1) \dots (k_r + 1)$

Proof: According to theorem 1.7, the positive divisors of n precisely those integers

$$d = p_1^{a1} p_2^{a2} \dots p_r^{ar}$$

Where $0 \leq a_i \leq k_i$.There are $k_1 + 1$ choices for the exponent $a_{1;} k_2 + 1$ $choices$ for a_2,…; and k_r+1 choices for a_r; Hence there are $(k_1 + 1)(k_2 + 1) \dots (k_r + 1)$ possible divisors of n.

In order to evaluate $\sigma(n)$,consider the product

$$(1 + p_1 + p_1^2 + \cdots + p_1^{k1})(1 + p_2 + p_2^2 \dots + p_2^{k2}) \dots (1 + p_r + p_r^2 + \cdots + p_r^{k1})$$

each positive divisor of n appears once and only once as a term in the expansion of the product, so that

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{k1}) \dots (1 + p_r + p_r^2 + \cdots + p_r^{kr}).$$

Applying the formula for the sum of a finite geometric series to the ith factor on the right -hand side, we get

$$1 + p_i + p_i^2 + \cdots + p_i^{ki} = \frac{p_i^{ki+1} - 1}{p_i - 1}$$

It follows that

$$\sigma(n) = \frac{p_1^{k1+1} - 1}{p_1 - 1} \frac{p_2^{k2+1} - 1}{p_2 - 1} \dots \frac{p_r^{kr+1} - 1}{p_r - 1}$$

Example 2.1.8: The number $180 = 2^2 . 3^2 . 5$ has

$180 = (2 + 1)(2 + 1)(1 + 1) = 18$ positive divisor. These are integers of the from

$$2^{a1} . 3^{a2} . 5^{a3},$$

Where $a_1 = 0,1,2;$ $a_2 = 0,1,2$ and $a_3 = 0,1$. Specifically, we obtain

$$1,2,3,4,5,6,9,10,12,15,18,20,30,36,45,60,90\ and 180$$

The sum of these integers is

$$\sigma(n) = \frac{2^3}{2-1}\frac{3^2}{3-1}\cdots\frac{5^2}{5-1} = \frac{7}{1}\frac{26}{2}\frac{24}{4} = 7.13.6 = 546.$$

Theorem 2.1.9[1]: The functions τ and σ are both multiplicative functions.

Proof. Let m and n be relatively prime integers. Because the result is trivially true if either m or n is equalto 1 we may assume that $m > 1$ and $n > 1$ if

$m = p_1^{k1}p_2^{k2}\dots p_r^{kr}$ and $n = q_1^{j1}q_2^{j2}\dots q_3^{j3}$ are the prime factorizations of rn and n, then because $gcd(m,n)$=1, no $p_i$can occur among the q_j. It follows that the prime factorization of the product $m\,n$ is given by $m\,n = p_1^{k1}\dots p_r^{kr}\ q_1^{j1}\dots q_r^{jr}$ .Applying to theorem 1.2 we obtain

$$\tau(m\,n) = [(k_1+1)\dots(k_r+1)[(j_1+1)\dots(j_s+1)]$$

$$= \tau(m)\tau(n)$$

In a similar fashion, Theorem 2.1.7 gives

$$\sigma(mn) = [\frac{p_1^{k1+1}-1}{p_1-1}\dots\frac{p_i^{ki+1}-1}{p_i-1}][\frac{q_1^{j1+1}-1}{q_1-1}\dots\frac{q_s^{ji+1}-1}{q_s-1}$$

$$= \sigma(m)\sigma(n)$$

Thus,$\tau\ and\ \sigma$ are multiplicative function.

Theorem 2.1.10[1] : (Mobius Inversion Formula)

Let F and f be two number – theoretic functions related by the formula

$$F(n) = \sum_{d|n} f(d) \text{ then}$$

F(n)=$\sum_{d|n}\mu(d)\,f\left(\frac{n}{d}\right) = \sum_{d|n}\mu(\frac{n}{d})f(d)$

Section Two

This section deals with that part of the theory arising out of the result known as Euler's Generalization of Fermat's Theorem

Definition 2.2.1[1] (Euler phi-function): For $n \geq 1$, let $\emptyset(n)$ denote the number of positive integers not exceeding n that are relatively prime to n.The function $\emptyset$ is usually called the Euler phi-function (sometimes, the indictor or totient)

Example 2.2.2: Let $n = 10$. Then find $\emptyset(n)$.

Solution: The integers do not exceed and relative prime to 10 are 1 ,3,7 and 9
Hence $\emptyset(n) = 4$.

Theorem 2.2.3: The function $\emptyset$ is a multiplicative function.

Theorem 2.2.4[1]: If p is a prime and $k > 0$, then

$$\emptyset(p^k) = p^k - p^{k-1} = p^k(1 - \frac{1}{p}).$$

Proof: clearly $gcd\ (n, p^k) = 1$ if and only if $p \nmid n$, there are p^{k-1} integers between 1 and p^k divisible by p, namely, $p, 2p, 3p \dots (p^{k-1})p$.

Thus, the set $\{1,2, \dots p^k\}$ contains exactly p^k–p^{k-1}integer that are relatively prime to p^k, and so by Definition 2.2.1, $\emptyset(p^k)$=p^k–p^{k-1}.

Example 2.2.5: we have

$$\emptyset(9) = \emptyset(3^2) = 3^2 - 3 = 6$$

The six integers less than and relatively prim being 1,2,4,5,7 and 8. To give a second illustration, there are 8 integers that are less than 16 and relatively prime to it; they are 1,3,5,7,9,11,13 and 15. Theorem 2.2.3 yields the same count:

$$\emptyset(16) = \emptyset(24) = 2^4 - 2^3 = 16 - 8 = 8$$

Theorem 2.2.6[1]: if the integer $n > 1$ has prime factorization

$n = p_1^{k1} p_2^{k2} \ldots p_r^{kr}$, then

$$\emptyset(n) = \left(p_1^{k1} - p_1^{k1-1}\right)\left(p_2^{k2} - p_2^{k2-1}\right) \ldots (p_r^{kr} - p_r^{kr-1})$$

$$=(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_r})$$

Proof: We intend to use induction on r, the number of distinct prime factor of n. By Theorem 2.2.3, the result is true for $r = 1$. Suppose it holds for $r = i$. Because

$$gcd(p_1^{k1} p_2^{k2} \ldots p_i^{ki}, p_{i+1}^{ki+1}) = 1$$

The Definition 1.10 gives

$$\emptyset\left(\left(p_1^{k1} \ldots p_i^{ki}\right)p_i^{ki+1}\right) = \emptyset\left(p_1^{k1} \ldots p_i^{ki}\right)\emptyset\left(p_{i+1}^{ki+1}\right)$$

$$= \emptyset(p_1^{k1} \ldots p_i^{ki})(p_{i+1}^{ki+1} - p_{i+1}^{ki+1-1})$$

Invoking the induction assumption, the first factor on the right-hand side become

$$\emptyset\left(p_1^{k1} p_2^{k2} \ldots p_i^{ki}\right) = (\, p_1^{k1} - p_1^{k1-1})(p_2^{k2} - p_2^{k2-1}) \ldots (p_i^{ki} - p_i^{ki-1})$$

and this serves to complete the induction step, and the proof.

Example 2.2.6: Let $m = 360$. Then calculate the value $\emptyset(360)$.

Solution: The prime factorization of 360 is $2^3.3^2.5$ and by Theorem 2.2.6

$$\emptyset(360) = 360(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})$$

$$= 360.\frac{1}{2}.\frac{2}{3}.\frac{4}{5}$$

$$= 96.$$

The next theorem show $\emptyset(n)$ is even integer.

Theorem 2.2.7[1]: For $n > 2$ $\emptyset(n)$ even integer.

Proof: First, assume that n is a power of 2, let us say that $n = 2^k$,with $k \geq 2$. By Theorem 2.2.6,

$\emptyset(n) = \emptyset(2^k) = 2^k\left(1 - \frac{1}{2}\right) = 2^{k-1}$, is an even integer.

If n dose not happen to be a power of 2, then it is divisible by an odd prime p, we therefore may write $n\ as\ n$=$p^k m$, where k ≥1 and $gcd(p^k, m) = 1$

Exploiting the multiplicative nature of the phi-function, we obtain

$$\emptyset(n) = \emptyset(p^k)\emptyset(m) = p^{k-1} = p - 1(p-1)\emptyset(m)$$

which again is even because $2|p$-1.

Lemma 2.2.8[1]: Let $n > 1$ and $gcd(a,n) = 1$ if $a_1, a_2 .. a_{\emptyset(n)}$ are the positive integer less than n and relatively prime to n, then

$$aa_1, aa_2 \ldots aa_{\emptyset(n)}$$

are congruent modulo n to $a_1, a_2 \ldots a_{\emptyset(n)}$ in some order.

Corollary 2.2.10[1] (Fermat): if p is a prime and $p \nmid a$, then $a^{p-1} \equiv 1(mod\ p)$

Example: Let $n = 3^{256}$ by using Euler theorem 2.2.9

Solution: To city a typical example, let us find the least two digits in the decimal representation of 3^{265} ; The is equivalent to obtaining the smallest nonnegative integer to which 3^{265} is congruent modulo 100.

Since

gcd (3,100) =1 and

$$\emptyset(100) = \emptyset(2^5.5^2)$$

$$\equiv 100\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right)(mod 100)$$

=40

Euler theorem yields

$$3^{40} \equiv 1(mod 100)$$

By the Division Algorithm, 256=6.40+16; whence

$$3^{256} \equiv 3^{6.40+16}(mod\ 100)$$

$$\equiv (3^{40})3^{16}(mod 100) = 3^{16}$$

and our problem reduces to one of evaluating 3^{16} , modulo 100 The calculation are as follows, with reasons omitted:

$$3^{16} \equiv (81)^{16}(mod\ 100)$$

$$\equiv (-19)^4(mod\ 100)$$

$$\equiv (361)^2(mod\ 100)$$

$$\equiv (61)^2(mod\ 100)$$

$$\equiv 21(mod\ 100)$$

There is another path to Euler theorem, one which requires the use of Fermat theorem

Section Two: Multiplicative Functions of Two Variable

In the section we discuss the notion of multiplicative function of two variable and we study some properties of multiplication function

Definition2.2.1 [3]: A function $f(m_1n_1, m_2n_2)$=$f(m_1, m_2)f(n_1, n_2)$for any $m_1, m_2, n_1, n_2 \in N$ such that $gcd(m_1, m_2, n_1, n_2) = 1$

Definition 2.2.2[3] :(Firmly Multiplicative Function) A Function f of two variables that is, f: N X N ⟶Z is firmly multiplicative if it is not identical zero and $f(m_1, n_1, m_2, n_2)$=$f(m_1, m_2)$ $f(n_1, n_2)$ holds forever m_1, n_1, m_2, n_2 ∈N such that

$gcd(m_1, n_1) g\ c\ d(m_2, n_2) = 1$. Let F_2 denoted the set of farmly multiplctivre functions of two variables.

Remark 2.2.3[3]: Clearly if f firmly multiplicative functions of two variables.

Proposition 2.2.4.[3]: A function $f \in F_2$ of two variable is firmly multiplicative if and only if there exist multiplicative function $f_1, f_2 \in M$ (M_1 the set of multiplicity function of one variables) $f(n_1, n_2)$=$f_1(n_1)f_2(n_2)$ for each $n_1, n_2 \in$ N. In the case f_1(n)=$f(n, 1)$and $f_2(n) = (1, n)$.

Example 2.2.6.[3]: The function (n_1, n_2)⟶ $\boldsymbol{\tau}$ (n_1) $\boldsymbol{\sigma}$ (n_2) and firmly multiplication.

Example 2.2.7[3]: For $n_1, n_2 \in N$ defined by

$$\alpha(n_1, n_2) = \begin{cases} 1, if\ n_1, n_2 are\ pirwis\ relatively\ prime \\ 0 \qquad\qquad\qquad\qquad otherwise \end{cases}$$

This function is multiplicative, which follows from the definition and for every $n_1, n_2 \in N$

$$a\ (n_1, n_2) = \sum_{d1|n1, d2|n2} \tau(d_1, d_2)\mu(n_1|d_1)\mu(n_2|d_2).$$

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الملخص

في هذا العمل، نناقش فكرة الخرائط المدمجة بحيث يتم تغييرها أو تغييرها كخريطة.

كما أخذ الموبيان وخريطة أويلر-فاي أمثلة على تلك الخرائط

پوختە

لەم ئیشەدا ئێمە گفتوگۆ دەکەین دەربارەی بیرۆکەی نەخشە لێکدراوەکان بۆیە گۆراو یان دوو گۆراو وەک نەخشەی مۆبیەنس و نەخشەی ئۆیلەر-فای هەروەها نموونەمان وەرگرتوە بۆ ئەونەخشانە.

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