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On Quasi Elements In Group Ring Z_nG

Research Project

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By

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Certification of the Supervisors

I certify that this work was prepared under my supervision at the Department of Mathematics/ College of Education /Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of

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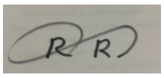
Signature : 

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Content

Abstract

In this work we study and discuss the concept of quasi elements in the group ring Z_nG, where G is a cyclic group of order m.

List of symbols

Symbols	Descriptions
$gcd(a, b)$	Greatest Common Divisors Between a and b
$\forall$	For all
$\in$	Belong to
ϕ	Euler's Phi-Function
Z_n	The Ring of Integers modulo n
Z_nG	Group Ring
$a \equiv b(mod\ n)$	a is congruent to b modulo n
$\|G\|$	Order of G

Introduction

The study of numbers has always occupied a unique position in the world of mathematics. It may very well be the best subject for a student trying to learn what constitutes a mathematical proof, and to construct the proofs, such as the theories of congruences and prime numbers. Most of the results of this work can be considered as an application of the number theory.

The present work consists of two chapters along with a list of references at the end. The first chapter deals with some definitions and theorems about ring theory and number theory, which are needed in our work.

Chapter two includes two sections. In section one we study quasi elements in rings. In section two we introduce and study the concept of quasi element in the group ring Z_2G, where G is a cyclic group of order n. It is shown that in the group ring Z_2G, where G is a cyclic group of order $2n$ generated by g, the element $g + g^2 + g^3 + g^4 + g^5 + g^6 + \cdots + g^{2n-1}$ is a quasi element, but the element $1 + g + g^2 + g^3 + g^4 + g^5 + g^6 + \cdots + g^{2n-1}$ is not a quasi element, and we prove that in the group ring Z_2G , where G is a cyclic group of order $2^k n$, $k \geq 2$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}, \alpha_i \geq 0$, generated g, the element $g + g^3 + g^5 + \cdots + g^{2^k n-1}$ and $1 + g^2 + g^4 + g^6 + \cdots + g^{2^k n-2}$ are not a quasi element and $1 + g + g^3 + g^5 + \cdots + g^{2^k n-1}$ is a quasi element of Z_2G.

Chapter one

Background

In this chapter we take some known definitions and results that we are needed in our work.

Definition 1.1: (M.Burton 2010) Let a and b be given integers, with at least one of them different from zero. The greatest common divisor of a and b, denoted by $gcd(a,b)$, is the positive integer d satisfying the following:

(a) $d|a$ and $d|b$.

(b) If $c|a$ and $c|b$, then $c \leq d$.

Example 1.2: Let $a = 8$ and $b = 4$ then $gcd(8,4) = 4$.

Definition 1.3: (B.Fraleigh 2003) Let R be a ring with unity. An element a in R is a unit of R if it has a multiplicative inverse in R.

Example 1.4: Let $\mathbb{R}$ be a ring of real numbers. Then $2 \in \mathbb{R}$ is a unit, since $\frac{1}{2}$ is the inverse of 2, Then $2.\frac{1}{2} = 1$.

Definition 1.5: (M.Burton 2010) Let n be a fixed positive integer. Two integers a and b are said to be congruent modulo n, symbolized by

$$a \equiv b \pmod{n}$$

If n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Example 1.6: Let $n = 7$, then $24 \equiv 3 \pmod{7}$.

Definition 1.7: (B.Fraleigh 2003) If two nonzero elements of a ring R such that $a . b = 0$ then a and b are divisors of zero (or zero divisors).

Example 1.8: Let Z_{10} be a ring. Then 2 and 5 are not zero and $2.5 \equiv 0 \ (mod\ 10)$ then a and b are divisors of zero.

Definition 1.9: (B.Fraleigh 2003) An element a of a ring R is idempotent if $a^2 = a$.

Example 1.10: Let Z_6 be a ring and $3 \in Z_6$. Then $3^2 \equiv 3 \ (mod\ 6)$.

Definition 1.11: (C.Hung 2010) An element a of a ring R is called an m-idempotent, where $a^m = a$ and $a^l \neq a$, for each $l < m$.

Definition 1.12: (M.Burton 2010) For $n \geq 1$, let $\phi(n)$ denote the number of positive integers not exceeding n that are relatively prime to n.

Example 1.13: Let $n = 30.$ Then $\phi(30) = 8$ for among the positive integers that do not exceed 30, there are eight that are relatively prime to 30, specifically 1,7,11,13,17,19,23,29.

If n is a prime number, then every integer less than n is relatively prime to it, whence, $\phi(n) = n - 1$. For Example $\phi(7) = 6$.

Theorem 1.14: (M.Burton 2010) (Euler) If $n \geq 1$ and $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \ (mod\ n)$.

Example 1.15: Let $n = 7$ and $a = 5.$ Then $gcd(7,5) = 1$ and $\phi(7) = 6$. Hence by Theorem 1.15,

$$5^6 \equiv 1 \ (mod\ 7).$$

Definition 1.16: (Kandasamy 2002) An element a in R is said to be a quasi element ,if for every b in R there exist $\alpha > 1$ (α is a positive integer) such that $ab = a^{\alpha}b$.

Example 1.17: Let $n = 5$. Then the ring Z_5 is a quasi commutative ring .

Solution: $0.1 \equiv 0^5.1(mod\ 5)$ $1.1 \equiv 1^5.1(mod\ 5)$

$0.2 \equiv 0^5.2(mod\ 5)$ $1.2 \equiv 1^5.2(mod\ 5)$

$0.3 \equiv 0^5.3(mod\ 5)$ $1.3 \equiv 1^5.3(mod\ 5)$

$0.4 \equiv 0^5.4(mod\ 5)$ $1.4 \equiv 1^5.4(mod\ 5)$

$2.1 \equiv 2^5.1(mod\ 5)$ $3.1 \equiv 3^5.1(mod\ 5)$

$2.2 \equiv 2^5.2(mod\ 5)$ $3.2 \equiv 3^5.2(mod\ 5)$

$2.3 \equiv 2^5.3(mod\ 5)$ $3.3 \equiv 3^5.3(mod\ 5)$

$2.4 \equiv 2^5.4(mod\ 5)$ $3.4 \equiv 3^5.4(mod\ 5)$

$4.1 \equiv 4^5.1(mod\ 5)$ $4.2 \equiv 4^5.2(mod\ 5)$

$4.3 \equiv 4^5.3(mod\ 5)$ $4.4 \equiv 4^5.4(mod\ 5)$

Definition 1.18: (Kandasamy 2002) Let R be a commutative ring with unit 1 and G be a multiplicative group. The group ring RG of the group G over the ring R consists of all finite formal sums of the form $\sum_i \alpha_i g_i$ (i-runs over a finite number) where $\alpha_i \in R\ and\ g_i \in G$ satisfying the following condition:

i. $\sum_{i=1}^{n} \alpha_i\, g_i = \sum_{i=1}^{n} \beta_i\, g_i \Leftrightarrow \alpha_i = \beta_i\ \ for\ \ i = 1,2, \dots, n, g_i \epsilon G.$
ii. $(\sum_{i=1}^{n} \alpha_i\, g_i) + (\sum_{i=1}^{n} \beta_i\, g_i) = \sum_{i=1}^{n} (\alpha_i + \beta_i)\, g_i\ ;\ g_i \epsilon G.$
iii. $(\sum_i \alpha_i\, g_i)\big(\sum_j \beta_i\, g_i\big) = \sum_k \gamma_k m_k\ \ where\ \gamma_k = \sum \alpha_i \beta_i\ \ , g_i b_j = m_k\ .$
iv. $r_i m_i = m_i r_i\ \ for\ all\ r_i \in R\ and\ m_i \in G.$
v. $r \sum_{i=1}^{n} r_i g_i = \sum_{i=1}^{n} (r r_i)\, g_i\ for\ \ r_i, r \in R\ and\ \sum r_i g_i \in RG.$

RG is a ring with $0 \in R$ as its additive identity. Since $1 \in R$ we have $G = 1.G \subset G$ and $Re = R \subseteq RG$ where e is the identity of G. Clearly if we replace the group G by a semi group S we say RS is the semi group ring of the semi group S over the ring R.

Example 1.19: Let $Z_2 = \{0,1\}$ be the ring and $G = \langle g | g^3 = 1 \rangle$ then the group ring, $Z_2G = \{0, 1, g, g^2, 1+g, 1+g^2, g+g^2, 1+g+g^2\}$ and the order of group ring is $|Z_2G| = |Z_2|^{|G|} = 2^3 = 8$.

Chapter two

Section one

On Quasi Elements in Rings

In this chapter we study the elements in a ring R that ar quasi elements or not.

Proposition 2.1.1: If a is a unit element in a ring R such that $a^2 = 1$, then a is a quasi element .

Proof: Suppose $a^2 = 1$, then

$$a^3b = a^2ab = ab$$

Hence a is a quasi element of a ring R for $\alpha = 3$.

Proposition 2.1.2: If a is an idempotent element of a ring R, then a is a quasi element .

Proof: Suppose that a is an idempotent element of R, then by Definition 1.9, $a^2 = a$. and for each $b \in R$, then

$$a^2b = ab$$

Hence a is a quasi element of R for $\alpha = 2$.

Proposition 2.1.3: If a is m-idempotent element, then a is a quasi element.

Proof: Suppose a is an m-idempotent element, then by Definition 1.11, $a^m = a$, and for each $b \in R$

$$a^{m+1}b = ab$$

Hence a is a quasi element of R for $\alpha = m + 1$.

Remark 2.1.4: If $a, b \in R$ are zero divisors such that $ab = 0$, then $a^2b = 0$.

Lemma 2.1.5: If $a^2 = 0$, then a is not a quasi element.

Since for a unit $b \in R$ then $a^2b = 0b = 0$, but $ab \neq 0$ because b is a unit.

Section two

On Quasi Elements in $Z_n G$

Proposition 2.2.1: In the group ring $Z_p G$, $p > 2$ where G is a cyclic group of order 2 generated by g, the element ag^k is a quasi element if a is a unit.

Proof: Since p is a prime, we have $\phi(p) = p - 1$ is even,

Now $(ag^k)^{\phi(p)} = a^{\phi(p)}.(g^k)^{\phi(p)}$

$= 1.(g^{p-1})^k$ by Theorem 1.15

$= 1.(g^{2s})^k$

$= (g^2)^{sk}$

$= 1$

Proposition 2.2.2: In the group ring $Z_2 G$, where G is a cyclic group of order m generated by g, the element g^n for $1 \leq n \leq m - 1$ is a quasi element.

Proof: Let $a = g^n$ and $b \in Z_2 G$, then

$$a^{m+1}b = (g^n)^{m+1}b$$

$$= (g^n)^m.g^n.b$$

$$= (g^m)^n.g^n.b$$

$$= 1^n.g^n.b$$

Hence $a^{m+1}b = ab$, then a is a quasi element of $Z_2 G$.

Proposition 2.2.3: In the group ring $Z_2 G$, where G is a cyclic group of order $2n$ generated by g, the element $g + g^2 + g^3 + g^4 + g^5 + g^6 + \cdots + g^{2n-1}$ is a quasi element, but the element $1 + g + g^2 + g^3 + g^4 + g^5 + g^6 + \cdots + g^{2n-1}$ is not a quasi element.

Proof: First we prove to the case

$a = g + g^2 + g^3 + g^4 + g^5 + g^6 + \cdots + g^{2n-1}$, we see that

$$a^2 = \begin{pmatrix} g^2 & g^3 & g^4 & g^5 & g^6 & g^7 & g^8 & \cdots & g^{2n-1} & 1 \\ g^3 & g^4 & g^5 & g^6 & g^7 & g^8 & g^9 & \cdots & 1 & g \\ g^4 & g^5 & g^6 & g^7 & g^8 & g^9 & g^{10} & \cdots & g & g^2 \\ g^5 & g^6 & g^7 & g^8 & g^9 & g^{10} & g^{11} & \cdots & g^2 & g^3 \\ g^6 & g^7 & g^8 & g^9 & g^{10} & g^{11} & g^{12} & \cdots & g^3 & g^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boxed{1} & g & g^2 & g^3 & g^4 & g^5 & g^6 & \cdots & \cdots & g^{2n-2} \end{pmatrix} = 1$$

by Proposition 2.1.1, the element a is a quasi element of Z_2G.

Second time we prove to the case $a = 1 + g + g^2 + g^3 + g^4 + g^5 + \cdots + g^{2n-1}$, we see that

$$a^2 = \begin{pmatrix} 1 & g & g^2 & g^3 & g^4 & g^5 & g^6 & g^7 & \cdots & g^{2n-2} & g^{2n-1} \\ g & g^2 & g^3 & g^4 & g^5 & g^6 & g^7 & g^8 & \cdots & g^{2n-1} & 1 \\ g^2 & g^3 & g^4 & g^5 & g^6 & g^7 & g^8 & g^9 & \cdots & 1 & g \\ g^3 & g^4 & g^5 & g^6 & g^7 & g^8 & g^9 & g^{10} & \cdots & g & g^2 \\ g^4 & g^5 & g^6 & g^7 & g^8 & g^9 & g^{10} & g^{11} & \cdots & g^2 & g^3 \\ g^6 & g^6 & g^7 & g^8 & g^9 & g^{10} & g^{11} & g^{12} & \cdots & g^3 & g^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{2n-1} & 1 & g & g^2 & g^3 & g^4 & g^5 & g^6 & \cdots & \cdots & g^{2n-2} \end{pmatrix} = 0$$

Then by Lemma 2.1.5, the element a is not a quasi element of Z_2G.

Example 2.2.4: Let Z_2G be a group ring where G is a cyclic group of order 8 generated by $g,$ and $a = g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7$. Then

$$a^2 = \begin{pmatrix} g^2 & g^3 & g^4 & g^5 & g^6 & g^7 & 1 \\ g^3 & g^4 & g^5 & g^6 & g^7 & 1 & g \\ g^4 & g^5 & g^6 & g^7 & 1 & g & g^2 \\ g^5 & g^6 & g^7 & 1 & g & g^2 & g^3 \\ g^6 & g^7 & 1 & g & g^2 & g^3 & g^4 \\ g^7 & 1 & g & g^2 & g^3 & g^4 & g^5 \\ \boxed{1} & g & g^2 & g^3 & g^4 & g^5 & g^6 \end{pmatrix} = 1.$$

Hence by Proposition 2.1.1, the element a is a quasi element of Z_2G

Proposition 2.2.5: In the group ring Z_2G, where G is a cyclic group of order $2n+1$ generated by g, has at least two quasi elements.

Proof: Let $a = g + g^2 + g^3 + g^4 + g^5 + \cdots + g^{2n}$

$$a^2 = \begin{pmatrix} \boxed{g^2} & g^3 & g^4 & g^5 & g^6 & \cdots & g^{2n-1} & g^{2n} & 1 \\ g^3 & g^4 & g^5 & g^6 & g^7 & \cdots & g^{2n} & 1 & g \\ \boxed{g^4} & g^5 & g^6 & g^7 & g^8 & \cdots & 1 & g & g^2 \\ g^5 & g^6 & g^7 & g^8 & g^9 & \cdots & g & g^2 & g^3 \\ \boxed{g^6} & g^7 & g^8 & g^9 & g^{10} & \cdots & g^2 & g^3 & g^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \boxed{g} & g^2 & \boxed{g^3} & g^4 & \boxed{g^5} & \cdots & g^{2n-3} & \boxed{g^{2n-2}} \end{pmatrix} = a$$

Then by Proposition 2.1.2, we get that the element a is a quasi element of Z_2G.

Let $a = 1 + g + g^2 + g^3 + g^4 + g^5 + \cdots + g^{2n}$. Then

$$a^2 = \begin{pmatrix} \boxed{1} & g & g^2 & g^3 & g^4 & g^5 & \cdots & g^{2n-2} & g^{2n-1} & g^{2n} \\ g & g^2 & g^3 & g^4 & g^5 & g^6 & \cdots & g^{2n-1} & g^{2n} & 1 \\ \boxed{g^2} & g^3 & g^4 & g^5 & g^6 & g^7 & \cdots & g^{2n} & 1 & g \\ g^3 & g^4 & g^5 & g^6 & g^7 & g^8 & \cdots & 1 & g & g^2 \\ \boxed{g^4} & g^5 & g^6 & g^7 & g^8 & g^9 & \cdots & g & g^2 & g^3 \\ g^5 & g^6 & g^7 & g^8 & g^9 & g^{10} & \cdots & g^2 & g^3 & g^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \boxed{g^{2n}} & 1 & \boxed{g} & g^2 & \boxed{g^3} & g^4 & \cdots & \boxed{g^{2n-4}} & g^{2n-3} & \boxed{g^{2n-2}} \end{pmatrix} = a$$

Then by Proposition 2.1.2, the element a is a quasi element of Z_2G.

Example 2.2.6: Let $Z_2\ G,$ be the group ring of order $|G| = 9$ and

$a = g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8$

$$a^2 = \begin{pmatrix} \boxed{g^2} & g^3 & g^4 & g^5 & g^6 & g^7 & g^8 & 1 \\ g^3 & g^4 & g^5 & g^6 & g^7 & g^8 & 1 & g \\ \boxed{g^4} & g^5 & g^6 & g^7 & g^8 & 1 & g & g^2 \\ g^5 & g^6 & g^7 & g^8 & 1 & g & g^2 & g^3 \\ \boxed{g^6} & g^7 & g^8 & 1 & g & g^2 & g^3 & g^4 \\ g^7 & g^8 & 1 & g & g^2 & g^3 & g^4 & g^5 \\ \boxed{g^8} & 1 & g & g^2 & g^3 & g^4 & g^5 & g^6 \\ 1 & \boxed{g} & g^2 & \boxed{g^3} & g^4 & \boxed{g^5} & g^6 & \boxed{g^7} \end{pmatrix}$$

$$= g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8.$$

Then by Proposition 2.1.2, we get the element a is a quasi element of Z_2G.

Proposition 2.2.7: In the group ring Z_2G , where G is a cyclic group of order $2n$ and n is an odd integer, generated by g, the element $g + g^3 + g^5 + \cdots + g^{2n-1}$ is a quasi element .

Proof: Let $a = g + g^3 + g^5 + \cdots + g^{2n-1}$

$$a^2 = \begin{pmatrix} \boxed{g^2} & g^4 & g^6 & g^8 & g^{10} & \cdots & g^{2n-2} & 1 \\ g^4 & g^6 & g^8 & g^{10} & g^{12} & \cdots & 1 & g^2 \\ \boxed{g^6} & g^8 & g^{10} & g^{12} & g^{14} & \cdots & g^2 & g^4 \\ g^8 & g^{10} & g^{12} & g^{14} & g^{16} & \cdots & g^4 & g^6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{2n-2} & 1 & g^2 & g^4 & g^6 & \cdots & g^{2n-6} & g^{2n-4} \\ \boxed{1} & g^2 & \boxed{g^4} & g^6 & \boxed{g^8} & \cdots & g^{2n-4} & \boxed{g^{2n-2}} \end{pmatrix}$$

$$= 1 + g^2 + g^4 + g^6 + g^8 + \cdots + g^{2n-2}$$

$$a^3 = \begin{pmatrix} \boxed{g} & g^3 & g^5 & g^7 & g^9 & \cdots & g^{2n-3} & g^{2n-1} \\ g^3 & g^5 & g^7 & g^9 & g^{11} & \cdots & g^{2n-1} & g \\ g^5 & g^7 & g^9 & g^{11} & g^{13} & \cdots & g & g^3 \\ \boxed{g^7} & g^9 & g^{11} & g^{13} & g^{15} & \cdots & g^3 & g^5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{2n-3} & g^{2n-1} & g & g^3 & g^5 & \cdots & g^{2n-7} & g^{2n-5} \\ \boxed{g^{2n-1}} & g & \boxed{g^3} & g^5 & \boxed{g^7} & \cdots & g^{2n-5} & \boxed{g^{2n-3}} \end{pmatrix}$$

$$= g + g^3 + g^5 + \cdots + g^{2n-1}$$

Therefore $a^3 = a,$ then $\forall\ b \in Z_2\ G,$ we have $ab = a^3 b$. Hence by Proposition 2.1.3, a is a quasi element of $Z_2 G$

Example 2.2.8: Let $Z_2 G$ be a group ring, where $|G| = 14$. Then for the element $a = g + g^3 + g^5 + g^7 + g^9 + g^{11} + g^{13}$, we have

$$a^2 = \begin{pmatrix} \boxed{g^2} & g^4 & g^6 & g^8 & g^{10} & g^{12} & 1 \\ g^4 & g^6 & g^8 & g^{10} & g^{12} & 1 & g^2 \\ \boxed{g^6} & g^8 & g^{10} & g^{12} & 1 & g^2 & g^4 \\ g^8 & g^{10} & g^{12} & 1 & g^2 & g^4 & g^6 \\ \boxed{g^{10}} & g^{12} & 1 & g^2 & g^4 & g^6 & g^8 \\ g^{12} & 1 & g^2 & g^4 & g^6 & g^8 & g^{10} \\ \boxed{1} & g^2 & \boxed{g^4} & g^6 & \boxed{g^8} & g^{10} & \boxed{g^{12}} \end{pmatrix}$$

$= 1 + g^2 + g^4 + g^6 + g^8 + g^{10} + g^{12}$, and

$$a^3 = \begin{pmatrix} \boxed{g} & g^3 & g^5 & g^7 & g^9 & g^{11} & g^{13} \\ g^3 & g^5 & g^7 & g^9 & g^{11} & g^{13} & g \\ \boxed{g^5} & g^7 & g^9 & g^{11} & g^{13} & g & g^3 \\ g^7 & g^9 & g^{11} & g^{13} & g & g^3 & g^5 \\ \boxed{g^9} & g^{11} & g^{13} & g & g^3 & g^5 & g^7 \\ g^{11} & g^{13} & g & g^3 & g^5 & g^7 & g^9 \\ \boxed{g^{13}} & g & \boxed{g^3} & g^5 & \boxed{g^7} & g^9 & \boxed{g^{11}} \end{pmatrix} = a.$$

Therefore $a^3 = a,$ then $\forall\ b \in Z_2\ G,$ we have $ab = a^3 b$. Hence by Proposition 2.1.3, a is a quasi element of $Z_2 G$.

Proposition2.2.9: In the group ring Z_2G, where G is a cyclic group of order $2^k n$, $k \geq 2$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}, \alpha_i \geq 0$, , generated by g, have elements that are not quasi element and have some elements that are quasi element.

Proof: Case1: Let $a = g + g^3 + g^5 + \cdots + g^{2^k n-1}$

$$a^2 = \begin{pmatrix} g^2 & g^4 & g^6 & g^8 & g^{10} & g^{12} & \cdots & g^{2^k n-2} & 1 \\ g^4 & g^6 & g^8 & g^{10} & g^{12} & g^{14} & \cdots & 1 & g^2 \\ g^6 & g^8 & g^{10} & g^{12} & g^{14} & g^{16} & \cdots & g^2 & g^4 \\ g^8 & g^{10} & g^{12} & g^{14} & g^{16} & g^{18} & \cdots & g^4 & g^6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{2^k n-2} & 1 & g^2 & g^4 & g^6 & g^8 & \cdots & g^{2^k n-6} & g^{2^k n-4} \\ 1 & g^2 & g^4 & g^6 & g^8 & g^{10} & \cdots & g^{2^k n-4} & g^{2^k n-2} \end{pmatrix} = 0$$

Then by Lemma 2.1.5, the element a is not a quasi element of Z_2G.

Case2: Let $a = 1 + g^2 + g^4 + g^6 + \cdots + g^{2^k n-2}$. Then

$$a^2 = \begin{pmatrix} 1 & g^2 & g^4 & g^6 & g^8 & \cdots & g^{2^k n-4} & g^{2^k n-2} \\ g^2 & g^4 & g^6 & g^8 & g^{10} & \cdots & g^{2^k n-2} & 1 \\ g^4 & g^6 & g^8 & g^{10} & g^{12} & \cdots & 1 & g^2 \\ g^6 & g^8 & g^{10} & g^{12} & g^{14} & \cdots & g^2 & g^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{2^k n-4} & g^{2^k n-2} & 1 & g^2 & g^4 & \cdots & g^{2^k n-8} & g^{2^k n-6} \\ g^{2^k n-2} & 1 & g^2 & g^4 & g^6 & \cdots & g^{2^k n-6} & g^{2^k n-4} \end{pmatrix} = 0,$$

by lemma 2.1.5 a is not quasi element of Z_2G.

Case3: let $a = 1 + g + g^3 + g^5 + \cdots + g^{2^k n-1}$. Then

$$a^2 = \begin{pmatrix} \boxed{1} & g & g^3 & g^5 & \cdots & g^{2^k n-3} & g^{2^k n-1} \\ g & g^2 & g^4 & g^6 & \cdots & g^{2^k n-2} & 1 \\ g^3 & g^4 & g^6 & g^8 & \cdots & 1 & g^2 \\ g^5 & g^6 & g^8 & g^{10} & \dots & g^2 & g^4 \\ g^7 & g^8 & g^{10} & g^{12} & \dots & g^4 & g^6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{2^k n-3} & g^{2^k n-2} & 1 & g^2 & \cdots & g^{2^k n-6} & g^{2^k n-4} \\ g^{2^k n-1} & 1 & g^2 & g^4 & \cdots & g^{2^k n-4} & g^{2^k n-2} \end{pmatrix} = 1, \text{ then}$$

By Proposition 2.1.1, the element a is a quasi element of Z_2G.

Case 4: Let $a = g^2 + g^4 + g^6 + \cdots + g^{2^k n-2}$, then

$$a^2 = \begin{pmatrix} g^4 & g^6 & g^8 & \cdots & g^{2^k n-2} & 1 \\ g^6 & g^8 & g^{10} & \cdots & 1 & g^2 \\ g^8 & g^{10} & g^{12} & \dots & g^2 & g^4 \\ g^{10} & g^{12} & g^{14} & \dots & g^4 & g^6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{2^k n-2} & 1 & g^2 & \cdots & g^{2^k n-8} & g^{2^k n-6} \\ \boxed{1} & g^2 & g^4 & \cdots & g^{2^k n-6} & g^{2^k n-4} \end{pmatrix} = 1, \text{ then}$$

by Proposition 2.1.1, the element a is a quasi element of Z_2G.

Example 2.2.10: Let Z_2G be the group ring, where $|G| = 2^2.3$

$a = g + g^3 + g^5 + g^7 + g^9 + g^{11}$.

$$a^2 = \begin{pmatrix} g^2 & g^4 & g^6 & g^8 & g^{10} & 1 \\ g^4 & g^6 & g^8 & g^{10} & 1 & g^2 \\ g^6 & g^8 & g^{10} & 1 & g^2 & g^4 \\ g^8 & g^{10} & 1 & g^2 & g^4 & g^6 \\ g^{10} & 1 & g^2 & g^4 & g^6 & g^8 \\ 1 & g^2 & g^4 & g^6 & g^8 & g^{10} \end{pmatrix} = 0$$

Then by Lemma 2.1.5 a is not quasi element of Z_2G.

Proposition 2.2.11: In the group ring Z_2G , where G is a cyclic group of order $2n$, for n is an odd integer generated by g, has a quasi elements.

Proof: Let $a = 1 + g^2 + g^4 + g^6 + \cdots + g^{2n-2}$. Tthen

$$a^2 = \begin{pmatrix} \boxed{1} & g^2 & g^4 & g^6 & g^8 & \cdots & g^{2n-4} & g^{2n-2} \\ g^2 & g^4 & g^6 & g^8 & g^{10} & \cdots & g^{2n-2} & 1 \\ \boxed{g^4} & g^6 & g^8 & g^{10} & g^{12} & \cdots & 1 & g^2 \\ g^6 & g^8 & g^{10} & g^{12} & g^{14} & \cdots & g^2 & g^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boxed{g^{2n-2}} & 1 & \boxed{g^2} & g^4 & \boxed{g^6} & \cdots & g^{2n-6} & \boxed{g^{2n-4}} \end{pmatrix} = a$$

Therefore by proposition 2.1.2, the element a is a quasi element of Z_2G.

Let $a = g^2 + g^4 + g^6 + \cdots + g^{2n-2}$, then

$$a^2 = \begin{pmatrix} \boxed{g^4} & g^6 & g^8 & g^{10} & g^{12} & \cdots & g^{2n-2} & 1 \\ g^6 & g^8 & g^{10} & g^{12} & g^{14} & \cdots & 1 & g^2 \\ \boxed{g^8} & g^{10} & g^{12} & g^{14} & g^{16} & \cdots & g^2 & g^4 \\ g^{10} & g^{12} & g^{14} & g^{16} & g^{18} & \cdots & g^4 & g^6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \boxed{g^2} & g^4 & \boxed{g^6} & g^8 & \cdots & g^{2n-6} & \boxed{g^{2n-4}} \end{pmatrix} = a$$

$$= g^2 + g^4 + g^6 + \cdots + g^{2n-2}$$

Therefore by Proposition 2.1.2, a is quasi element of Z_2G.

Example 2.2.12: Let Z_2G , be the group ring, where $|G| = 14$. Then for the element $a = 1 + g^2 + g^4 + g^6 + g^8 + g^{10} + g^{12}$, we have

$$a^2 = \begin{pmatrix} \boxed{1} & g^2 & g^4 & g^6 & g^8 & g^{10} & g^{12} \\ g^2 & g^4 & g^6 & g^8 & g^{10} & g^{12} & 1 \\ \boxed{g^4} & g^6 & g^8 & g^{10} & g^{12} & 1 & g^2 \\ g^6 & g^8 & g^{10} & g^{12} & 1 & g^2 & g^4 \\ \boxed{g^8} & g^{10} & g^{12} & 1 & g^2 & g^4 & g^6 \\ g^{10} & g^{12} & 1 & g^2 & g^4 & g^6 & g^8 \\ \boxed{g^{12}} & 1 & \boxed{g^2} & g^4 & \boxed{g^6} & g^8 & \boxed{g^{10}} \end{pmatrix}$$

$= 1 + g^2 + g^4 + g^6 + g^8 + g^{10} + g^{12} = a$

Therefore by proposition 2.1.2, the element a is a quasi element of Z_2G.

Proposition 2.2.13: In the group ring Z_nG, where G is a cyclic group of order mn and n is odd , the element

$$1 + g^m + g^{2m} + g^{3m} + \cdots + g^{mn-4} = \sum_{k=0}^{mn-4} g^{km}$$

is a quasi element.

Proof: Let $a = 1 + g^m + g^{2m} + \cdots + g^{mn-4}$, then

$$a^2 = \begin{pmatrix} \boxed{1} & g^m & g^{2m} & \dots & g^{mn-8} & g^{mn-4} \\ g^m & g^{2m} & g^{3m} & \dots & g^{mn-4} & 1 \\ \boxed{g^{2m}} & g^{3m} & g^{4m} & \dots & 1 & g^m \\ g^{3m} & g^{4m} & g^{5m} & \dots & g^m & g^{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boxed{g^{mn-4}} & 1 & \boxed{g^m} & \dots & g^{mn-12} & \boxed{g^{mn-8}} \end{pmatrix} = a$$

oposition 2.1.2, a is a quasi element of Z_nG.

Example 2.2.14: In the group ring Z_2G , where $|G| = 20$, the element

$a = 1 + g^4 + g^8 + g^{12} + g^{16}$ is a quasi element, because

$$a^2 = \begin{pmatrix} \boxed{1} & g^4 & g^8 & g^{12} & g^{16} \\ g^4 & g^8 & g^{12} & g^{16} & 1 \\ \boxed{g^8} & g^{12} & g^{16} & 1 & g^4 \\ g^{12} & g^{16} & 1 & g^4 & g^8 \\ \boxed{g^{16}} & 1 & \boxed{g^4} & g^8 & \boxed{g^{12}} \end{pmatrix} = a, \text{ then}$$

by proposition 2.1.2, a is a quasi element of Z_2G.

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پوختە

ئامانجمان لەم کارە لێکۆڵینەوەو گفتوگۆکردنە دەربارەی بیرۆکەی دانەی پاشگری لە ئەڵقەی گروپەکان Z_nG ، کاتێ G گروپێکی خولیە بە ئۆردەری m .