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Notes on Clean Rings and Clean Elements

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Abstract. It is proved in this note that if R is an Abelian right pm-ring with a finite number of minimal prime ideals then R is a clean ring, which extends a main result in [1]. And some known results in [9] on clean elements of commutative reduced rings are extended to arbitrary Abelian rings. Also we give some new characterizations of a (strongly) clean ring.

Keywords: Exchange rings; Clean rings; Clean elements; Right pm-rings.

1. Introduction

Throughout this note R denotes an associative ring with identity. We use the symbol $U(R)$ to denote the group of units of R and $Id(R)$ the set of idempotents of R. The Jacobson radical, the prime radical of R are denoted by $J(R)$, $P(R)$, respectively. The symbol $Max(R)$ stands for the maximal spectrum of the ring R, and $Max_r(R)$ its right maximal spectrum.

Following Nicholson [8], an element x of a ring R is called to be clean if $x = u + f$ where $u \in U(R)$ and $f \in Id(R)$. And the ring R is clean if every element of R is clean. In [7], an element $x \in R$ is said to be strongly clean if $x = u + f$ with $u \in U(R)$, $f \in Id(R)$ and $uf = fu$. While the ring R is strongly clean if every element of R is strongly clean. We call an element x of R to be an exchange element if there exists $e \in Id(R)$ such that $e \in xR$ and $1-e \in (1-x)R$. This definition is left-right symmetric by the proof of [6, Proposition]. Clearly the ring R is an exchange ring if and only if every element of R is an exchange element (cf. [5]). It is known from Nicholson [5, Proposition 1.8] that a clean element of a ring R is an exchange element and the converse holds when R is Abelian.

According to Sun [10], a proper right ideal P of a ring R is prime if $aRb \subseteq P$ implies $a \in P$ or $b \in P$. And the ring R is said to be a right pm-ring if every right prime ideal is contained in a unique maximal right ideal. This is equivalent to saying that each prime ideal is contained in a unique maximal right ideal [10, p. 185]. By Sun [10, p. 184], if R is a right *pm*-ring then R is a *pm*-ring (the ring in which every prime ideal is contained in a unique maximal ideal). And for a right *pm*-ring R, $Max(R) = Max_r(R)$ ([10, p. 186]).

Recall that a ring R is reduced if it has no nonzero nilpotent element and a ring R is Abelian if all idempotents of R are central.

Motivated by the results of Anderson and Camillo [1] on commutative clean rings, Samei [9] on clean elements in commutative reduced rings. In this note we continue to study clean rings and clean elements, extending some known results of [1] and [9]. Also we give some new characterizations of a (strongly) clean ring.

2. Some Results on Abelian Clean Rings

In this section, we wish to extend two main results in [1].

We start with the following lemma which is essentially duo to Anderson and Camillo [1, Lemma 20]. The new proof is given here for convenience of the reader.

Lemma 2.1. *Let R be any ring. If $e, f \in Id(R)$ with $ef = fe$ and $e - f \in J(R)$, then $e = f$.*

Proof. Since $e - f \in J(R)$, $1 - e + f \in U(R)$. Let $1 - e + f = u$. Then $(1-e)u^{-1} + fu^{-1} = 1$. Hence $efu^{-1} = e$. Since $ef = fe$, we have $fu^{-1} = u^{-1}f$, which implies $e(1 - f) = 0$ and thus $e = ef$. Similarly $f = fe$ and so $e = f$ as desired. ∎

Theorem 2.2. *Let R be an Abelian ring with a finite number minimal prime ideals (e.g., R is Noetherian). Then the following conditions are equivalent.*

(1) *R is a finite direct product of local rings.*

(2) *R is a clean ring.*

(3) *R is a right pm-ring.*

Proof. (1)⇒ (2) Since a local ring is a clean ring and the cleanness is closed under the finite direct product, so R is clean.

(2)⇒ (3) It is known by [3, Theorem 1] that if R is an Abelian exchange ring then R/P is a local ring for every prime ideal P of R. Hence P is contained in a unique maximal right ideal of R. Note that a clean ring is an exchange ring, and so we are done.

(3)⇒ (1) Since R is a right *pm*-ring, it is a *pm*-ring. So each minimal prime ideal is contained in a unique maximal ideal. And it is well known that a maximal ideal is a prime ideal and every prime ideal contains a minimal prime ideal, hence

R has only finitely many maximal ideals $M_1, M_2, \cdots, M_n$. In the case of $n = 1$, R is a local ring since $Max(R) = Max_r(R)$, and we are done. Otherwise $n > 1$, we can assume that $M_i \neq M_j$ whenever $i \neq j$. Let $P_1, P_2, \cdots, P_s$ be all minimal prime ideals of R. Then clearly $s \geq n$. Assume $P_{t1}, P_{t2}, \cdots, P_{tk_t}$ be minimal prime ideals contained in $M_t, t = 1, 2, \cdots, n$ where $k_1 + k_2 + \cdots + k_n = s$. Let I_t be the intersection of minimal prime ideals contained in M_t. We claim that I_i and I_j are comaximal for $i \neq j$. If not, then $I_i + I_j \neq R$. There exists a maximal ideal M_k such that $I_i + I_j \subseteq M_k$. Hence $I_i, I_j \subseteq M_k$. Since $I_t = P_{t1} \bigcap P_{t2} \bigcap \cdots \bigcap Ptk_t$, and $P_{t1} P_{t2} \cdots P_{tk_t} \subseteq I_t$ for each $1 \leq t \leq n$, there exist minimal prime ideals P_{im}, P_{jl} such that $P_{im}, P_{jl} \subseteq M_k$. Note that $P_{im} \subseteq M_i$ and $P_{jl} \subseteq M_j$. It yields that $k = i = j$ since R is a *pm*-ring, which is a contradiction. Thus I_i and I_j are comaximal. Next we claim that R/I_t is local for each t. If not, then R/I_t contains at least two maximal right ideals and hence I_t is contained in two maximal right ideals, say, M'_1, M'_2 which are two sided maximal ideals since R is a *pm*-ring. It yields that $I_t \subseteq M'_1 \bigcap M'_2$. So there exists a minimal prime ideal P_{tk} such that $P_{tk} \subseteq M'_1 \bigcap M'_2$, which contradicts the fact that R is a *pm*-ring. Now clearly we have $P(R) = I_1 \bigcap I_2 \bigcap \cdots \bigcap I_n$. By the Chinese Remainder Theorem, $R/P(R) = R/I_1 \bigoplus R/I_2 \bigoplus \cdots \bigoplus R/I_n$. Hence $R/P(R)$ is an Abelian clean ring, so is R by [8, Proposition 6]. Since $R/P(R)$ is obviously orthogonally finite, so is R. In fact, if R is orthogonally infinite then there exist infinite orthogonal idempotents $e_1, e_2, \cdots$ such that $e_i \neq e_j$ whenever $i \neq j$. Since $R/P(R)$ is orthogonally finite, there must be $i \neq j$ such that $\bar{e_i} = \bar{e_j}$. It follows that $e_i - e_j \in P(R) \subseteq J(R)$, and so $e_i = e_j$ by Lemma 2.1, a contradiction. Since R is orthogonally finite, R is a semiperfect ring by [2, Theorem 9]. Hence there exist orthogonal local idempotents $e_1, e_2, \cdots, e_q$ such that $e_1 + e_2 + \cdots + e_q = 1$ in R. So $R = e_1 R \bigoplus e_2 R \bigoplus \cdots \bigoplus e_q R$ where every $e_l R$ is a local ring. Thus the proof is completed. ■

As an immediate result of the above theorem, we have the following corollary.

Corollary 2.3. *([1, Theorem 5]) Let R be a commutative ring with a finite number minimal primal ideals (e.g., R is Noetherian) then the following conditions are equivalent.*

(1) *R is a finite direct product of local rings.*

(2) *R is a clean ring.*

(3) *R is a pm-ring.*

We conclude this section with the following theorem, which extends Theorem 14 in [1].

Theorem 2.4. *Let R be a ring. Then $R = U(R) \bigcup Id(R)$ if and only if R is a division ring or a Boolean ring.*

Proof. ($\Rightarrow$) Obviously R is reduced and hence Abelian. If R contains only trivial idempotents, then $R = U(R)\bigcup\{0\}$ and so R is a division ring. Otherwise there exists $e \in Id(R)$ such that $e \neq 0, 1$. We claim that $2 = 0$ in R. In fact, $2 \in Id(R)$ certainly implies $2 = 0$. If $2 \in U(R)$, then we have $2e \in Id(R)$. Assume to the contrary, then $2e \in U(R)$ and so $e \in U(R)$, which gives $e = 1$, a contradiction. Since $2e \in Id(R)$, $2e = 0$. It yields that $2 = 2e + 2(1-e) = 2(1-e) \in U(R)$. Thus $1 - e \in U(R)$ and so $e = 0$, again a contradiction. Hence $2 = 0$. It follows that $Id(R)$ is a subring of R. We claim that $Id(R) = R$. If not, then there exists $r \in U(R)$ but $r \notin Id(R)$. Note that $er \in U(R)$ implies $e \in U(R)$, and so $e = 1$, which is impossible. Hence we have $er \in Id(R)$. Similarly $(1-e)r \in Id(R)$. Therefore $r = er + (1-e)r \in Id(R)$, a contradiction. Hence $R = Id(R)$, and R is a Boolean ring.

($\Leftarrow$) Clear. ∎

Corollary 2.5. *([1, Theorem 14]) Let R be a commutative ring. Then $R = U(R)\bigcup Id(R)$ if and only if R is a field or a Boolean ring.*

3. Clean Elements and Clean Rings

We start this section with the following lemma which is essentially duo to Nicholson [5, Proposition 1.8].

Lemma 3.1. *Let R be an Abelian ring. If $x \in R$ is an exchange element, then x is a clean element.*

Proof. To check the proof of [5, Proposition 1.8(2)] case by case. ∎

Lemma 3.2. *([7, Proposition 3]) Let R be a ring and $e \in Id(R)$. If $a \in eRe$ is strongly clean in eRe, then a is strongly clean in R.*

Recall that a subset B of a ring R is called to be clean if every element of B is clean (cf. [9, p. 3480]).

Proposition 3.3. *Let R be an Abelian ring and $x \in R$. If x^n is clean, then x is clean. In particular, if I is clean then $\sqrt{I}$ is clean for any ideal of R.*

Proof. First we prove that x is an exchange element. Since x^n is a clean element, there exist $u \in U(R)$ and $f \in Id(R)$ such that $x^n = u + f$. Let $e = u(1-f)u^{-1}$. Then $(x^n - e)u = (u+f)u - u(1-f) = x^{2n} - x^n$, so $e = x^n + (x^n - x^{2n})u^{-1}$. Now it is easy to check that $e \in xR$ and $1 - e \in (1-x)R$, so x is an exchange element. Hence x is a clean element by Lemma 3.1. The last assertion follows from the fact that $\sqrt{I} \subseteq \{a \in R \mid a^n \in I \ for\ some\ n \geq 1\}$. ∎

Corollary 3.4. *Let R be an Abelian ring. If a^2 is clean, then a and $-a$ are clean.*

It should be note that Proposition 3.3 and Corollary 3.4 are obtained in [9, Corollary 2.4] only for commutative reduced rings.

The following proposition extends Proposition 2.5 in [9].

Proposition 3.5. *Let R be an Abelian ring. Let $a \in R$ be clean in R and $e \in Id(R)$. Then we have*

(1) *ae is clean.*

(2) *If $-a$ is also clean, then $a + e$ is clean.*

Proof. (1) Since a is clean in R, $a = u + f$ with $u \in U(R)$ and $f \in Id(R)$. So $ae = ue + fe$. Clearly $ue \in U(eRe)$ and $fe \in Id(eRe)$ since R is Abelian. And R is Abelian implies ae is strongly in eRe. By Lemma 3.2, ae is strongly clean in R and hence ae is clean.

(2) It is known and easy to prove that a is clean if and only if $1 - a$ is clean for $a \in R$. Since a and $-a$ are clean, so are a and $1 + a$. Let $a = u + f$ and $1+a = v+g$ where $u, v \in U(R)$ and $f, g \in Id(R)$. Then $a+e = ae+a(1-e)+e = (1 + a)e + a(1 - e) = (v + g)e + (u + f)(1 - e) = ve + u(1 - e) + ge + f(1 - e)$. Note that R is Abelian, it is easy to check that $ve + u(1 - e) \in U(R)$ with $(ve + u(1 - e))^{-1} = v^{-1}e + u^{-1}(1 - e)$ and $ge + f(1 - e) \in Id(R)$. Hence $a + e$ is clean in R. ∎

Corollary 3.6. *([9, Proposition 2.5]) Let R be commutative reduced ring. Let $a \in R$ be clean and $e \in R$ be idempotent, then*

(1) *ae is clean.*

(2) *If $-a$ is also clean, then $a + e$ is clean.*

We conclude this note by giving some new characterizations of a (strongly) clean ring.

Recall that a ring R is an exchange ring if and only if for every $x \in R$ there exist $e \in Id(R)$ and $a, b \in R$ such that $e = xa$ and $e - 1 = (x - 1)b$. And it is known that there exists an exchange ring R which is not clean [2]. So we naturally ask that what forms of the above a and b have for a clean ring R. The following theorem answer the question.

Theorem 3.7. *Let R be a ring. Then R is clean if and only if for every $x \in R$ there exist $e \in Id(R), w \in U(R)$ such that $e = xwe$ and $e - 1 = (x - 1)w(e - 1)$.*

Proof. Proof. Suppose R is clean and $x \in R$. Then $x = u + f$ with $u \in U(R), f \in Id(R)$. Let $e = u(1-f)u^{-1}$. Then $(x-e)u = (u+f-u(1-f)u^{-1})u = u^2 + fu - u + uf = x^2 - x$. So $e - x = (x - x^2)u^{-1}$, which implies $e = x + x(1-x)u^{-1} = x(1+(1-x)u^{-1})$. Note that $1 - x = 1 - u - f$ and $u^{-1}e = (1-f)u^{-1}$. So we have $e = x(1+(1-u-f)u^{-1}) = x(1-f)u^{-1} = xu^{-1}e$. Since $u^{-1}e = (1-f)u^{-1}$, we have $fu^{-1} = u^{-1}(1-e)$. Now $e = x + x(1-x)u^{-1}$ implies $e - 1 = (x-1)(1-xu^{-1})$. Hence $e - 1 = (x-1)(1-(u+f)u^{-1}) = (x-1)(-fu^{-1}) = (x-1)u^{-1}(e-1)$. Now take $w = u^{-1}$, then we are done.

Conversely, assume that for every $x \in R$ there exist $w \in U(R), e \in Id(R)$ such that $e = xwe$ and $e - 1 = (x-1)w(e-1)$. We prove that x is clean. In fact, $e - 1 = (x-1)w(e-1)$ implies $1 - e = -xwe + we + xw + -w$, so $1 - e = -e + we + xw - w$ since $e = xwe$. It follows that $xw = w + 1 - we$ and hence $x = w^{-1} + 1 - wew^{-1} = w^{-1} + w(1-e)w^{-1}$, as desired. ∎

Similarly we have the following corollary.

Corollary 3.8. *Let R be a ring. Then R is clean if and only if for every $x \in R$ there exist $e \in Id(R), w \in U(R)$ such that $e = ewx$ and $e - 1 = (e-1)w(x-1)$.*

Theorem 3.9. *Let R be any ring and $x \in R$. The following statements are equivalent.*

(1) *x is clean.*

(2) *There exist $e \in Id(R)$ and $u \in U(R)$ such that $e = uxe$ and $e - 1 = u(x-1)(e-1)$.*

(3) *There exist $e \in Id(R)$ and $u \in U(R)$ such that $e = exu$ and $e - 1 = (e-1)(x-1)u$.*

Proof. (1)⇒(2) If $x = v + f$ with $v \in U(R)$ and $f \in Id(R)$, then $xf = vf + f$. So $(x-1)f = vf$, which gives $f = v^{-1}(x-1)f$. From $x = v + f$, we have $x(f-1) = v(f-1)$. Hence $f - 1 = v^{-1}x(f-1)$. Now take $1 - e = f, u = v^{-1}$, then we have $e = uxe$ and $e - 1 = u(x-1)(e-1)$.

(2)⇒(1) Since $uxe = e$ and $e - 1 = u(x-1)(e-1)$ implies $e - 1 = uxe - ux - ue + u$, we have $ux + ue - u = 1$, which gives $x = u^{-1} + 1 - e$. Thus x is clean.

(1)⇔(3) Similar to the above proof. ∎

Corollary 3.10. *Let R be a ring and $x \in R$. Then x is strongly clean if and only if there exist $e \in Id(R), u \in U(R)$ such that $e = xue, e - 1 = (x-1)u(e-1)$ and $eu = ue$. Hence R is strongly clean if and only if every x of R satisfies the above conditions.*

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