Advanced Calculus

Infinite Sequence and Aeries:

OVERVIEW While everyone knows how to add together two numbers, or even several, how to add together infinitely many numbers is not so clear. In this chapter we study such questions, the subject of the theory of infinite series. Infinite series sometimes have a finite sum, as $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$. Other infinite series do not have a finite sum, as with $1+2+3+4+5+\cdots$

Sequence

A sequence is a list of numbers $a_1, a_2, a_3, \dots, a_n, \dots$ in a given order. Each of $a_1, a_2, a_{3,\dots}$ and so on represents a real number. These are the **term**s of the sequence.

For example the sequence $2,4,6,8, \dots$ has first term 2, second term 4 and nth term $2n$.

The integer n is called the **index** of a_n, and indicates where occurs in the list. We can think of the sequence as a function that sends 1 to a_1, 2 to a_2, 3 to a_3, and in general sends the positive integer n to the nth term a_n.

Definition: (Infinite Sequence)

An infinite sequence of numbers is a function whose domain is the set of positive integers.

The function associated to the sequence $12,14,16,18,20,22, \dots$ is described by the formula $a_n = 10 + 2n$. Sequence can be described by listing terms such as:

$$\{a_n\} = \{\sqrt{n},\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots\}$$

$$\{b_n\} = \{(-1)^n\} = \{1, -1, 1, -1, 1, -1, \dots\}$$

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence $\left\{\frac{1}{n}\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ whose terms approach 0 as n gets large. On the other hand, Sequences like $\{1,2,3,4,\ldots\}$ have terms that get larger than any number as n in increases, and sequences like $\{1,-1,1,-1,1,-1,\ldots\}$ bounce back and forth between 1 and -1 never converging to a single value.

The following definition captures the meaning of having a sequence converge to a limiting value. It says that if we go far enough out in the sequence, by taking the index n to be larger than some value N, the difference between a_n and the limit of the sequence becomes less than any selected number.

Definitions:(Converges, Diverges, Limit)

The sequence $\{a_n\}$ converges to the number L if to every positive number ε there corresponds an integer N such that: $n > N \Rightarrow |a_n - L| < \varepsilon$.

If no such number L exists, we say that $\{a_n\}$ diverges. If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the limit of the sequence.

Example:- Show that $\left\{\frac{1}{n}\right\}$ converges to 0.

Solution:

Example: Show that $\{k\}$ converges to k.

Solution:

Example: Show that the sequence $\{(-1)^n\}$ diverges

Solution:

Calculating Limits of Sequences

Theorem1: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = \text{A}$ and $\lim_{n\to\infty} b_n = B$, then

1. Sum Rule: $\{a_n + b_n\}$ Converges to $A + B$
2. Difference Rule: $\{a_n - b_n\}$ Converges to $A - B$
3. Product Rule $\{a_n . b_n\}$ Converges to $A.B$
4. Constant Multiple Rule: $\{k a_n\}$ Converges to kA
5. Quotient Rule: $\left\{\frac{a_n}{b_n}\right\}$ Converges to $\frac{A}{B}$, $B \neq 0$.

Theorem 2: (The Sandwich Theorem for Sequences)

Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of real number. If $\{a_n\} \le \{b_n\} \le \{C_n\}$ hold for all n and if $\{a_n\}$ and $\{ c_n \}$ Converges to L, then $\{b_n\}$ also Converge to L.

Example: Show that $\left\{\frac{\cos n}{n}\right\}$ converges to 0.

Solution:

Theorem 3: The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, Then $f(a_n) \to f(L)$.

Example: applying above theorem, Show that $\sqrt{\frac{n+1}{n}} \longrightarrow 1$

Solution:

Theorem 4: The following six sequences converge to the limits listed bellows:

1 . $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

2 . $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

3 . $\lim\limits_{n\to\infty} x^{1/n} = 1;\ x > 0$

4 . $\lim\limits_{n\to\infty} x^n = 0\ ; |x| < 1$

5 . $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ any x

6 . $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0\ ;$ any x

Example:

1) $\lim\limits_{n\to\infty} \sqrt[n]{n^2} =$

2) $\lim\limits_{n\to\infty} \left(-\frac{1}{2}\right)^n =$

3) $\lim\limits_{n\to\infty} \frac{\ln n^2}{n} =$

4) $\lim\limits_{n\to\infty} \frac{n-2}{n} =$

Theorem 5: A sequence $\{a_n\}$ converges to 0 if and only if the sequence of absolute values $\{|a_n|\}$ converges to 0.

Bounded Non-decreasing Sequences

The terms of a general sequence can bounce around, sometimes getting larger, sometimes smaller. An important special kind of sequences is one for which each term is at least as large as its predecessor.

Definition: Non-decreasing Sequence

A sequence $\{a_n\}$ with property that $a_n \le a_n + 1$ for all n is called a nondecreasing sequence.

Example: Show that $\left\{\frac{n^2}{n+1}\right\}$ is nondecreasing sequence.

Solution:

Definitions: (Bounded, Upper Bound, Least Upper Bound)

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $|a_n| \leq M$. The number M is an **upper bound** for $\{a_n\}$. If M is upper bound for $\{a_n\}$, but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

Example: Applying the definition for Boundedness

(a) The sequence $\{1,2,3, \dots .\}$ has no upper bound.

(b) The sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \dots\right\}$ is bounded above by1.

Theorem: (The Non-decreasing Sequence Theorem)

A non-decreasing sequence of real numbers converges if and only if it is bounded from above. If a non-decreasing sequence converges, if converges to its least upper bound.

Example: Use the non-decreasing sequence theorem to convergence the sequence $\left\{\frac{n}{n+1}\right\}$.

Solution:

Definition: (Non-increasing Sequence)

A sequence of numbers $\{a_n\}$ in which $a_n \geq a_{n+1}$ for every n is called a non-increasing sequence

Definition: (Bounded from Below and Lower Bounded)

A sequence $\{a_n\}$ is **bounded from below** if there is a number M with $M \leq a_n$ for every n, such a number M is called a **lower bound** for the sequence.

Example: Let $a_n = \frac{n+1}{n} = 2\,, \frac{3}{2}\,, \frac{4}{3}\,, \frac{5}{4}\,, \ldots$

The sequence $\{a_n\}$ is bounded above by 2 and bounded below by 1.

Example: Show that $\left\{\frac{n+1}{n}\right\}$ is non increasing sequence.

Solution:

Infinite Series

An infinite series is the sum of an infinite sequence of $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$. The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead we look at what we get by summing the first n terms of the sequence and stopping.

The sum of the first n terms $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is an ordinary finite sum and can be calculated by normal addition. It is called the nth partial sum. As n gets larger, we expect the partial sums to get closer and closer to limiting value in the same sense that the terms of a sequence approach a limit.

For example, to assign to an expression like $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

we add the terms one at a time from the beginning and look for a pattern in how these partial sums grow.

$S_1 = 1 = 2 - 1$

$S_2 = 1 + \dfrac{1}{2} = 2 - \dfrac{1}{2}$

$S_3 = 1 + \dfrac{1}{2} + \dfrac{1}{4} += 2 - \dfrac{1}{4}$

.

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$S_n = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^{n-1}} = 2 - \dfrac{1}{2^{n-1}}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \dfrac{1}{2^{n-1}} = 2$

Thus $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 2$

Definitions: (Infinite series, nth term, Partial Sum, Converges)

Given a sequence of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + \cdots + a_n + \cdots$ is an **infinite series**. The number a_n is the nth term of the series. The sequence that defined by $S_n = a_1 + a_2 + \cdots + a_n$ is **the sequence of partial sums** of the series, the number S_n being the $\boldsymbol{nth}$ **partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its sum is L. In this case, we write:

$$a_1 + a_2 + \cdots + a_n = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

Geometric series

are series of the form $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$

in which a and r are fixed real numbers and $a \neq 0$.

Theorem7:

1. If $|r| < 1$,then the geometric series $ar + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$

converges to $\frac{a}{1-r}$.

2. $|r| \geq 1$ the series $ar + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Proof: $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$

$rS_n = ar + ar + ar^2 + ar^3 + \cdots + ar^{n+1}$ (Multiply by r)

$(1-r)S_n = a(1-r^{n+1})$ (Subtraction)

$$S_n = \frac{a(1-r^{n+1})}{(1-r)}$$

1. $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{(1-r)} = \frac{a}{1-r}$ if $|r| < 1$

2. $\lim_{n\to\infty} S_n$ does not exist if $|r| \geq 1$

Example: The geometric series $\sum_{n=1}^{\infty} \frac{1}{3^{n+1}} = \frac{1}{9} + \frac{1}{27} + \frac{1}{83}$ is converges.

Solution:

Example: The geometric series $\sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + \cdots$ is diverges.

Solution:

Example: Repeating Decimals

Express the repeating decimal 5232323... as the ratio of two integers

Solution:

H.W: Determine the Geometric series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4}$ is converges or diverges.

Combining Series

Whenever we have to convergent series, we can add them term by subtract them by term, or multiply them by constant to make new convergent series.

Theorem 8: If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

1. Sum Rule: $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(a_n + b_n)$
2. Difference Rules: $\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(a_n - b_n)$
3. Constant Multiple Rule: $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n$

proof: The three rules for series follow from the analogous rules for sequences in Theorem 1, To prove sum Rule for series, let

$A_n = a_1 + a_2 + \cdots + a_n$ and

$B_n = b_1 + b_2 + \cdots + b_n$

Then the partial sum of $\sum_{n=1}^{\infty}(a_n + b_n)$ is

$S_n = (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)$

The sequence $S_n = A_n + B_n = a_1 + \cdots + a_n + b_1 + \cdots + b_n$ ...

Converges to A + B

Hence $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(a_n + b_n) = A + B$

Telescoping Series:

Example: Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$.

Solution:

H.W. 1. $\sum_{n=1}^{\infty} \frac{5}{n(n+2)}$

Theorem 9: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem 8 leads to a test for detecting the kind of divergence.

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Example: Applying the nth-Term Test

a. $\sum_{n=1}^{\infty} n^2$ diverges, because
b. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges, because
c. $\sum_{n=1}^{\infty} (-1)^n$ diverges, because
d. $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges, because

Theorem 10: (The Integral Test)

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$ where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Proof: We establish the test for the case N$= 1$. The proof for general N is similar.

We start with the assumption that f is a decreasing function with $a_n = f(n)$ for every n. This leads us to observe that the rectangles in (Fig A) which have areas $a_1, a_2, \dots, a_n$ collectively enclose more area than that under the curve from $x = 1$ to $x = n + 1$. That is

$$\int_1^{n+1} f(x)dx \leq a_1 + a_2 + \cdots + a_n \quad \text{(upper integral)}$$

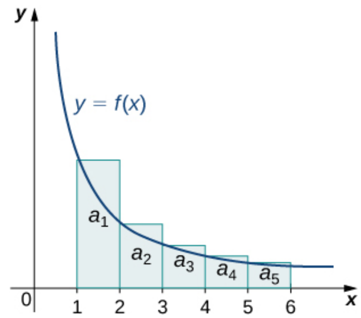


Fig A

In (Fig B) the rectangles have been faced to the left instead of to the right. If we mentality disregard the first rectangle, of area a_1, we see that

$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)dx$ (lower integral)

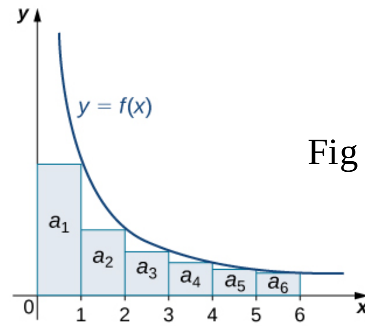


Fig B

If we include, we have

$$\int_1^{n+1} f(x)dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)dx$$

This inequalities hold for each n, and continue to hold as $n \to \infty$.

If $\int_1^\infty f(x)dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite. If $\int_1^\infty f(x)dx$ is infinite, then the left-hand inequality shows that $\sum a_n$ is infinite. Hence the seires and the integral are both infinite.

Definition:(P –series) are series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, in which p is a real constant.

Example: Applying the Integral Test

Shows that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (pa real constant) converges if $p > 1$ and diverges if $p \le 1$.

Solution:

Example: By the integral test show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is converge.

Solution:

Comparison Test:

We have seen how to determine the convergence of geometric series, p-series, and a few others. We can test the convergence of many more series by combining their terms to those of a series whose convergence is known.

Theorem 11: (The Comparison Test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with no negative terms.

a) $\sum_{n=1}^{\infty} a_n$ converges if there is a convergent series $\sum_{n=1}^{\infty} b_n$ with $a_n \leq b_n$ for all $n > N$ for some integer N.
b) $\sum_{n=1}^{\infty} a_n$ diverges if there is a divergent series $\sum_{n=1}^{\infty} b_n$ with $a_n \geq b_n$ for all $n > N$ for some integer N.

Example: The series $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges, because

The Limit Comparison Test

We now introduce a comparison test that is particulary useful for series in which is a rational function of n.

Theorem 12: (Limit Comparison Test)

Suppose that $a_n > 0$ and $b_n > 0$ for all $n > N(N$ an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converges or both diverges.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: we will prove part1

Since $\frac{c}{2} > 0$, there exists an integer N, such that for all n

$$n > N \Rightarrow \left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}$$ {limit definition with $\varepsilon = \frac{c}{2}, L = c$

$$n > N \Rightarrow -\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}$$ *and* a_n *replaced by* $\frac{a_n}{b_n}$}

$$n > N \Rightarrow c - \frac{c}{2} < \frac{a_n}{b_n} < c + \frac{c}{2}$$

$$n > N \Rightarrow \left(\frac{c}{2}\right) b_n < a_n < \frac{3c}{2} b_n$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} \frac{3c}{2} b_n$ converges and $\sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test. If $\sum_{n=1}^{\infty} b_n$ diverges, then$\sum_{n=1}^{\infty} \frac{c}{2} b_n$ diverges and $\sum_{n=1}^{\infty} a_n$ diverges by the Direct Comparison Test.

Example: The series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is converges, because

Example: Using the Limit Comparison Test

Which of the following series converge and which diverge?

$a)\quad \frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\cdots=\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$

$b)\quad \frac{1}{1}+\frac{1}{3}+\frac{1}{7}+\cdots=\sum_{n=1}^{\infty}\frac{1}{2^n-1}$

$c)\quad \frac{1+2\ln 2}{9}+\frac{1+3\ln 3}{14}+\cdots=\sum_{n=1}^{\infty}\frac{1+n\ln n}{n^2+5}$

Solution:

The Ratio and Root Test

The Ratio Test measures the rate of growth (or decline) of a series by examining the ratio $\frac{a_{n+1}}{an}$. For a geometric series Σar^{n-1}, this rate is a constant ($\frac{ar^{n+1}}{ar^n} = r$), and the series converges if and only if its ratio is less than 1 in absolute value. The Ratio Test is a powerful rule extending that result. We prove it on the next page using the Comparison Test.

Theorem: (The Ratio Test)

Let $\sum a_n$ be a series with positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = P$. Then

(a) The series converges if $P < 1$,

(b) The series diverges if $P > 1$ or P is infinite,

(c) The test is inconclusive if $p = 1$.

Proof: (a) $P < 1$. Let r be a number between 1 and P.

Then the number $\varepsilon = r - p$ is positive. Since

$$\frac{a_{n+1}}{a_n} \to p,$$

$\frac{a_{n+1}}{a_n}$ must lie within ε of P when n is large enough, say for all $n \geq N$. In particular

$$\frac{a_{n+1}}{a_n} < p + \varepsilon = r, \quad \text{where } n \geq N$$

That is

$$a_{n+1} < ra_n,$$

$$a_{N+1} < ra_N ,$$

$$a_{N+2} < ra_{N+1} < r^2 a_{N,}$$

$$a_{N+3} < ra_{N+2} < r^3 a_{N,}$$

$$\vdots$$

$$a_{N+m} < a_{N+m-1} < r^m a_{N\,.}$$

These inequalities show that the terms of our series, after the Nth term, approach zero more rapidly than the terms in a geometric series with ratio $r < 1$. More precisely, consider the series $\sum c_n$, where $c_n = a_n$ for all n, and

$$\sum_{n=1}^{\infty} c_n = a_1 + a_2 + \cdots + a_{N-1} + a_N + ra_N + r^2 a_N + \cdots$$

$$= a_1 + a_2 + \cdots + a_{N-1} + a_N(1 + r + r^2 + \cdots)$$

The geometric series $1 + r + r^2 + \cdots$ converges because $|r| < 1$, so $\sum c_n$ converges. Since $a_n \leq c_n,\ \sum a_n$ also converges.

(b) $1 < p \leq \infty$. From some index M on,

$$\frac{a_{n+1}}{a_n} > 1 \qquad \text{and} \qquad a_M < a_{M+1} < a_{M+2} < \cdots$$

The terms of the series do not approach zero as n because infinite and the series diverges by the nth -Term Test.

$(\boldsymbol{c})$ $P = 1$. The two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Show that some other test for convergence must be used when $p = 1$.

For $\sum_{n=1}^{\infty} \frac{1}{n}$: $\frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1.$

For $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1.$

In both cases, $p = 1$, yet the first series diverges, whereas the second converge.

Example: Applying the Ratio Test

Investigate the convergence of the following series

$$(a)\ \sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} \qquad (b)\ \sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!} \qquad (c)\ \sum_{n=1}^{\infty} \frac{4^n n!\,n!}{(2n)!}$$

Solution:

Theorem: (The Root Test)

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$ and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = P.$$

Then

a) the series converges if $P < 1$,

b) the series diverges if $P > 1$ or P is infinite

c) the series inconclusive if $P = 1$.

Example: which of the following series converges and which diverges?

$$(a)\sum_{n=1}^{\infty}\frac{n^2}{2^n} \qquad (b)\sum_{n=1}^{\infty}\frac{2^n}{n^2} \qquad (c)\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$$

Solution:

Alternating Series, Absolute and Condition Convergence

A series in which the terms are alternately positive and negative is an alternating series.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\frac{(-1)^{n+1}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}4}{2^{n}}+\cdots$$

$$1-2+3-4+5-6+\cdots+(-1)^{n+1}n+\cdots$$

Theorem: **(The Alternating Series Test (Leibniz's) Theorem)**

The series $\sum_{n=1}^{\infty}(-1)^{n}u_{n}=u_{1}-u_{2}+u_{3}-u_{4}+\cdots$ converges if all three of the following conditions are satisfied:

1. the $u_n{}'s$ are all positive.
2. $u_n \geq u_n+1$ for all $n \geq N$, for some positive integer N.
3. $u_n \to 0$.

Example: Are the alternating harmonic series

(1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ and (2) $\sum_{n=1}^{\infty}\frac{(-1)^{n}4}{2^{n}}$ are converges or diverges?

Solution:

Definition: (Absolutely Convergent)

A series $\sum a_n$ **converge absolutely** (is absolutely convergent) if the corresponding series of absolute values $\sum|a_n|$ converges.

Example: The geometric series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^{n-1}}$ converges absolutely because the corresponding series of absolute values $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ converges.

Definition: (Conditionally Convergent)

A series that converges but does not converge absolutely its **converges conditionally**.

Example: Conditionally Convergent

Is the series $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n+3}$ conditionally convergent?

Solution:

Theorem: (The Absolute Convergence Test)

If $\sum_{n=1}^{\infty} | a_n |$ converges, then $\sum_{n=1}^{\infty} a_n$ Converges.

Proof: For each n,

$-|a_n| \leq a_n \leq |a_n|$, so $0 \leq a_n + | a_n| \leq 2|a_n|$.

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n |$ converges and by the Direct Comparison Test, the nonnegative series $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n| .$$

Therefore $\sum_{n=1}^{\infty} a_n$ converges.

Example: Is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges?

Solution:

Power Series

Now that we can test infinite series for convergence. We can study the infinite polynomials mentioned at the beginning of this chapter. We call this polynomials power series because they are defined as infinite series of some variable, in our case x. Like polynomials, power series can be added subtracted, multiplicities differentiated and integrated to give new power series.

Definitions: Power series, Centre, coefficients

A **power series about $\boldsymbol{x = 0}$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \qquad (1)$$

A **power series about $\boldsymbol{x = a}$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

In which the **center $\boldsymbol{a}$** and the **coefficients** $c_{0,}c_{1,} \dots, c_{n,} \dots$ are constants.

Example: Taking all the coefficients to be 1 in Equation (1) gives the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

This is the geometric series with first term 1 and ratio x, it converges to $\frac{1}{1-x}$ for $|x| < 1$. We express this fact by writing

Example: Testing for convergence using the Ratio Test

for what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\, x^2 + 3!\, x^3 + \cdots$

Solution

Theorem: (The Convergence Theorem for Power Series)

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

Corollary to Theorem: The convergence of the series $\sum_{n=0}^{\infty} c_n (x - a)^n$ is described by one of the following three possiblities:

1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges absolutely for every x $(R = \infty)$.

3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

Term-by-Term Differentiation

A theorem from advanced calculus says that a power series can be differentiated term by term at each interior point of its interval of convergence.

Theorem: (The Term-by-Term Differentiation Theorem)

If $\sum c_n\,(x-a)^n$ converges for $a-R<x<a+R$ for some $R>0$, It defines a function:

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \text{ on } a-R<x<a+R.$$

Such a function f has derivatives of all orders inside the interval of convergence.

We can obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n-1}^{\infty} nc_n\,(x-a)^{n-1}.$$

$$f''(x)=\sum_{n=2}^{\infty} n\,(n-1)c_n(x-a)^{\,n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

Example: Find series for $f'(x)$ and $f''(x)$ if

$$f(x)=\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots+x^n+\cdots$$

$$=\sum_{n=0}^{\infty} x^n, \qquad -1<x<1.$$

Solution:

Theorem: The Term-by-Term Integration Theorem

Suppose that $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ Converges for $a-R<x<a+R$ $(R>0)$. Then

$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$, Converges for $a-R<x<a+R$ and

$$\int f(x)\,dx=\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}+c$$

For $a-R<x<a+R$.

Example: A series $\tan^{-1} x$, $-1\le x\le 1$ identify the function

$$f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots,-1\le x\le 1.$$

Solution:

Example: A series for $\ln(1+x)$, $-1<x\le 1$ is the series $\frac{1}{1+t}=1-t+t^2-t^3+\cdots$

Solution:

Theorem : (The Series Multiplication Theorem for Power Series)

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

Then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Example : Multiply the geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ \ldots + x^n + \ \ldots = \frac{1}{1-x} \text{ ,for } |x| < 1,$$

by itself to get a power series for $\frac{1}{(1-x)^2}$,for $|x| < 1$

Solution:

Taylor and Maclaurin Series

This section shows how functions that are infinitely differentiable generate power series called **Taylor series**. In many cases, these series can provide polynomial approximations of the generating functions.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n + \ldots$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I we obtain

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \ldots$$

$$f''(x) = 1.2\, a_2 + 2.3\, a_3(x-a) + 3.4\, a_4(x-a)^2 + \ldots$$

$$f'''(x) = 1.2.3\, a_3 + 2.3.4\, a_4(x-a) + 3.4.5\, a_5(x-a)^2 + \ldots$$

with the *nth* derivative , for all n , being

$f^{(n)}(x) = n!\ a_n + a\ sum\ of\ terms\ with\ (x-a)$ as a factor.

Since these equations all hold at $x = a$, we have

$$f'(a) = a_1$$

$$f''(a) = 1.2\, a_2$$

$$f'''(a) = 1.2.3\, a_3$$

and in general

$$f^{(n)}(a) = n!\ a_n$$

$$a_n = \frac{f^{(n)}(a)}{n!}$$

If f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Definitions: (Taylor Series, Maclaurin Series)

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

the Taylor Series generated by f at $x = 0$.

Convergence of Taylor Series

Theorem: (Taylor's Theorem)

If f and its first n derivative $f', f'', \ldots, f^n$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)\,(b-a) + \frac{f''(a)}{2!}\,(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Taylor's Formula:

If f has derivatives of all orders in an open interval containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}\,(x-a)^n + R_n(x)$$

Where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}$ for some c between a and (x).

If $lim_{n\to\infty} R_n(x) = 0$ for all x we say that the Taylor series generated by f at $x = a$ converges to f on I.

Example: Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\frac{1}{x}$?

Solution:

Example: Find the Taylor series and the Taylor polynomials by $f(x) = e^x$ at $x = 0$.

Solution:

Example: Show that (H.W.)

1. $sin\, x = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
2. $cos\, x = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad , \ |x| \le 1$

The Binomial Series

The Taylor series generated by $f(x) = (1+x)^m$, where m is constant, is

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ \ldots + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k + \ \ldots .(1)$$

This series called the binomial series, converges absolutely for $|x| < 1$. To derive the series, we first list the function and its derivatives:

$$f(x) = (1+x)^m$$

$$f'(x) = m(1+x)^{m-1}$$

$$f''(x) = m(m-1)(1+x)^{m-2}$$

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$$

$$\vdots$$

$$f^k(x) = m(m-1)(m-2)\ldots(m-k+1)(1+x)^{m-k}$$

we then evaluate these at $x = 0$ and substitute into the Taylor series formula to obtain Series (1)

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k}\ x^k$$

Where we define

$$\binom{m}{1} = m \,, \binom{m}{2} = \frac{m(m-1)}{2!} \text{ and } \binom{m}{k} = \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!} \qquad for\ \ k \ge 3$$

Example: Evaluating the limits by using power series: $lim_{x\to0}(\frac{1}{sin\,x} - \frac{1}{x})$

Solution:

Frequently used Taylor series

1. $\frac{1}{1-x} = 1 + x + x^2 + \ ... + \ x^n + \cdots = \sum_{n=0}^{\infty} x^n \, , \ |x| < 1$
2. $\frac{1}{1+x} = 1 - x + x^2 - \ ... + \ (-x)^n + \cdots = \sum_{n=0}^{\infty}(-1)^n \, x^n \, , \quad |x| < 1$
3. $e^x = 1 + x + \frac{x^2}{2!} + \ ... + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$
4. $sin\,x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}, |x| < \infty$
5. $cos\,x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n}}{(2n)!}, \quad |x| < \infty$
6. $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \, x^n}{n}, \quad -1 < x \le 1$
7. $ln\frac{1+x}{1-x} = 2\tanh^{-1} x = \ 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$
8. $tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{2n+1}, \quad |x| \le 1$

Binomial series

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \cdots + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!} x^k + \ \ldots$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k \ , \quad |x| < 1 \ ,$$

Where

$$\binom{m}{1} = m \ , \binom{m}{k} = \frac{m(m-1)}{2!} \ , \binom{m}{k} = \frac{m(m-1)\ldots(m-k+1)}{k!} \ , \text{for } k \geq 3 \ .$$

Fourier series

Suppose we wish to approximate. A function f on the interval $[0, 2\pi]$ by a sum of Sine and cosine function,

$$f_n(x) = a_\circ + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx)$$

$$= a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx\ ) \quad \ldots\ldots\ldots(1)$$

We would like to choose values for the constants $a_0, a_1, a_2, \ldots, a_n$ *and* $b_0, b_1, b_2, \ldots, b_n$ that make $f_n(x)$ an "best possible" approximation to $f(x)$. The notion of "best possible" is defined as follows:

1- $f_n(x)$ and $f(x)$ give the same value when integrated from 0 to 2π.

2- $f_n(x) \cos kx$ and $f(x) \cos kx$ give the same value when integrate from *0 to* $2\pi(k = 1, \ldots, n)$.

3- $f_n(x) \sin kx$ and $f(x) \sin kx$ give the Same. value integrated from *0 to* $2\pi(k = 1, \ldots, n)$.

we chose f_n so that the integrals on the left remain the same when f_n is replaced by f,so we can use these equations to find $a_0, a_1, \ldots a_n$ and $b_0, b, \ldots b_n$ from f:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx \quad \ldots..(2)$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \ dx \ldots (3)$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \ \sin kx \ dx \ldots (4)$$

The only condition needed to find these coefficients is that the integrals above must exist. If we let $n \to \infty$ and use these rules to get the coefficients of on infinite series, then the resulting sum is called the **Fourier series** for $f(x)$,

$$a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

Example: Finding a Fourier series Expansion Fourier series can be used to represent some functions that cannot be represented by Taylor series; for example

$$f(x) = \begin{cases} 1, & if \quad 0 \leq x \leq \pi \\ 2, & if \quad \pi < x \leq 2\pi \end{cases}$$

Solution:

Functions of several variables:

Definitions: (Function of n independent variables)

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$A **real-valued function** f on D is a rule that assigns a unique (single) real number $w = f(x_1, x_2, \dots, x_n)$ to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** $\boldsymbol{x_1}$ to x_nWe also call the x_j's the function's **input variables** and call w the function's **output variable.**

Example: Find the value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $p(3,0,4)$.

Solution:

Example: Find the domain and range of the following function :

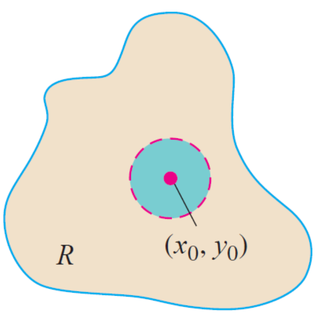
(a) $w = \sqrt{y - x^2}$
(b) $w = \frac{1}{xy}$
(c) $w = \sin xy$

Solution :

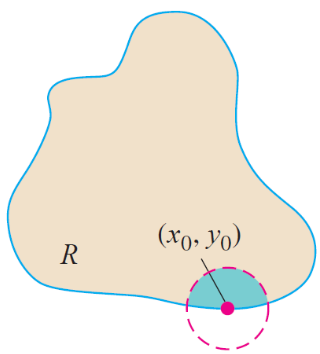
Definition: Interior and Boundary Points, Open, Closed

A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R , A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points .

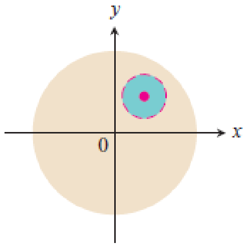


(a) Interior point

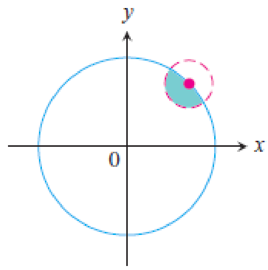


(b) Boundary point

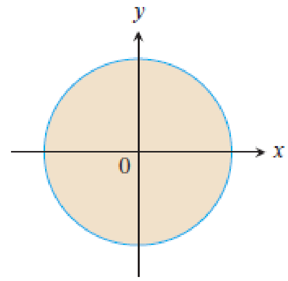
Example:



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an
interior point.



$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit
disk. (The unit
circle.)



$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.
Contains all
boundary points.

Definition: Bounded and Unbounded Regions in the Plane

A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Example: Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Solution:

Definition: (Level Curve, Graph, Surface)

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Example: Graphing a Function of Two Variables

Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$ and $f(x, y) = 75$ in the domain of f in the plane.

Solution:

Definition: (Limit of a Function of Two Variables)

We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0) and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$|f(x,y) - L| < \varepsilon$ < whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

Example:

Theorem: (Properties of Limits of Functions of Two Variables)

The following rules hold if L, M and k are real numbers and

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$.

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)}\big(f(x,y) + g(x,y)\big) = L + M$

2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)}\big(f(x,y) - g(x,y)\big) = L - M$

3. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)}\big(f(x,y).g(x,y)\big) = L.M$

4. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)}\big(kf(x,y)\big) = kL$

5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)}\left(\frac{f(X,y)}{g(x,y)}\right) = \frac{L}{M}$; $M \neq 0$

6. *Power Rule:* If *r* and *s* are integers with no common factors, and $S \neq 0$ then

$\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y))^{r/s} = L^{r/s}$,provided $L^{r/s}$ is a real number.(if s is even we assume that L > 0.

Example: Calculating limits

A) $\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5x-y^3} =$

B) $\lim\limits_{(x,y)\to(1,4)} \sqrt{x^2+y^2} =$

C) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} =$

Definition (Continuous Function of Two Variables)

A function f(x,y) is **continuous at the point** (x_0,y_0) if:

1. *f* is defined at (x_0,y_0).

2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \text{f}(x_0,y_0)$

A function is **continuous** if it is continuous at every point of its domain.

Example:

Show that $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$,is continuous at every point except the origin .

Solution:

Two-path test for Nonexistence of a limit

If a function $f(x, y)$ has different limits along two different paths as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Example: Show that the function $f(x, y) = \frac{2x^2y}{x^4+y^2}$ has no limit as (x, y) approaches (0, 0).

Solution:

Example: Define $f(0,0)$ in any way that extends f to be continuous at the origin, where

$$f(x,y) = \frac{3x^2y}{x^2+y^2}$$

Solution:

Partial Derivatives

Definition: (Partial Derivative with repeat to x)

The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\frac{\partial f}{\partial x}\Big|_{(x_o,y_o)} = \lim_{h\to 0}\frac{f(x_o+h, y_o) - f(x_o, y_o)}{h} = f_x$$

Provided the limit exists.

Definition: (Partial Derivative with repeat to y)

The partial derivative of $f(x, y)$ with respect to x at the point (x, y) is

$$\frac{\partial f}{\partial y}\Big|_{(x_o,y_o)} = \lim_{h\to 0}\frac{f(x_o, y_o+h) - f(x_o, y_o)}{h} = f_y$$

Provided the limit exists.

Example: Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if $f(x, y) = x^2 + 3xy + y - 1$

Example: Find f_x and f_y if

1. $f_{(x,y)} = \frac{2y}{y+\cos x}$ 2. $f(x, y) = \frac{x}{x^2+y^2}$ 3. $f(x, y) = e^{xy}\ln xy$

Higher Order of Partial Derivatives

When we differentiate a function $f(x, y)$ more than one times, we produce its as follows:

1. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = f_{xx}$

2. $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = f_{yy}$

3. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = f_{yx}$

4. $\dfrac{\partial f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = f_{xy}$

5. $\dfrac{\partial^3 f}{\partial x \partial y \partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 f}{\partial y \partial x}\right) = f_{xyx}$

6. $\dfrac{\partial^3 f}{\partial y \partial x^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 f}{\partial x^2}\right) = f_{xxy}$

$\vdots$

Example: If $f(x, y) = x \cos y + y\, e^{x}$, find

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^3 f}{\partial y \partial y \partial x}, \frac{\partial^3 f}{\partial x \partial x \partial x}$$

Solution:

Theorem: (The Mixed Derivative Theorem)

If $f(x,y)$ and its partial derivative f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a,b) and are all continuous at (a,b) then $f_{xy}(a,b) = f_{yx}(a,b)$

Theorem: (Differentiability Implies Continuity)

If a function $f(x,y)$ is differentiable at (x_o, y_o), then f is continuous at (x_o, y_o).

Example: Verify that $w_{yx} = w_{xy}$ at any point, where

$$w = e^x + x\ln y + y\ln x$$

Solution:

Definition: (Differentiable Function)

A function $z = f(x, y)$ is **differentiable at** $(\boldsymbol{x_o}, \boldsymbol{y_o})$ if $f_x(x_o, y_o)$ and $f_y(x_o, y_o)$ exist and Δz satisfies an equation of the form.

$\Delta z = f_x(x_o, y_o)\Delta x + f_y(x_o, y_o)\Delta y + \mathcal{E}_1\Delta x + \mathcal{E}_2\Delta y,$ in which each of $\mathcal{E}_1, \mathcal{E}_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if its differentiable at every point in its domain.

Theorem: (Chain Rule for Functions of two Independent Variables)

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t), y = y(t)$ are differentiable function of t, then the composite $w = f\big(x(t), y(t)\big)$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x\big(x(t), y(t)\big)x'(t) + f_y\big(x(t), y(t)\big)y'(t).$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$$

Proof: The proof consists of showing that if x and y are differentiable at $t = t_o$, then w is differentiable at t_o and

$$\left(\frac{dw}{dt}\right)_{t_o} = \left(\frac{\partial w}{\partial x}\right)_{p_o}\left(\frac{dx}{dt}\right)_{t_o} + \left(\frac{\partial w}{\partial y}\right)_{p_o}\left(\frac{dy}{dt}\right)_{t_o},$$

Where $p_o = \big(x(t_0), y(t_o)\big)$ The subscripts indicate where each of the derivatives are to be evaluated.

Let $\Delta x, \Delta y$ and Δw be the increments that result from changing from t to $t_o + \Delta t$.

Since f is differentiable,

$$\Delta w = \left(\frac{\partial w}{\partial x}\right)_{p_o} \Delta x + \left(\frac{\partial w}{\partial y}\right)_{p_o} \Delta y + \mathcal{E}_1 \Delta x + \mathcal{E}_2 \Delta y,$$

Where $\mathcal{E}_1, \mathcal{E}_2 \to 0\ as\ \Delta x, \Delta y \to 0$. To find $\frac{dw}{dt}$, we divide this equation through by Δt and let Δt approach zero.

The division gives:

$$\frac{\Delta w}{\Delta t} = \left(\frac{\partial w}{\partial x}\right)_{p_o} \frac{\Delta x}{\Delta t} + \left(\frac{\partial w}{\partial y}\right)_{p_o} \frac{\Delta y}{\Delta t} + \mathcal{E}_1 \frac{\Delta x}{\Delta t} + \mathcal{E}_2 \frac{\Delta y}{\Delta t}$$

Letting Δt approach zero gives

$$\left(\frac{dw}{dt}\right)_{p_o} = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \left(\frac{\partial w}{\partial x}\right)_{p_o} \left(\frac{dx}{dt}\right)_{t_o} + \left(\frac{\partial w}{\partial y}\right)_{p_o} \left(\frac{dy}{dt}\right)_{t_o} + 0\left(\frac{dx}{dt}\right)_{t_o} + 0\left(\frac{dy}{dt}\right)_{t_o}$$

Example: Find $\frac{dw}{dt}$ where $f(x, y) = x^2 + y\ and\ x = \cos t\,, y = t^2$

Solution:

Example: Use the Chain Rule to find the derivative of $w = xy$. with respect to t, the path $x = \cos t, y = \sin t$. What is the derivative value at $t = \frac{\pi}{2}$?

Solution:

Remark: If $w = f(x, y)$, $x = g(r, s)$, $y = h(r, s)$ then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

Example: Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms for r and s, if $w = x^2 + y^2$, $x = r - s$, $y = r + s$.

Solution:

Theorem *: (A Formula for Implicit Differentiation)

Suppose that $F(x,y)$ is differentiable t of and that the equation $F(x,y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$, $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Proof : Let $F(x,y) = 0 \quad \rightarrow \quad F_x \frac{dx}{dt} + F_y \frac{dy}{dt} = 0$

$$F_y \frac{dy}{dt} = -F_x \frac{dx}{dt} \quad \rightarrow \quad \frac{dy/dt}{dx/dt} = -\frac{F_x}{F_y} \quad \rightarrow \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example: Use theorem * to find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$

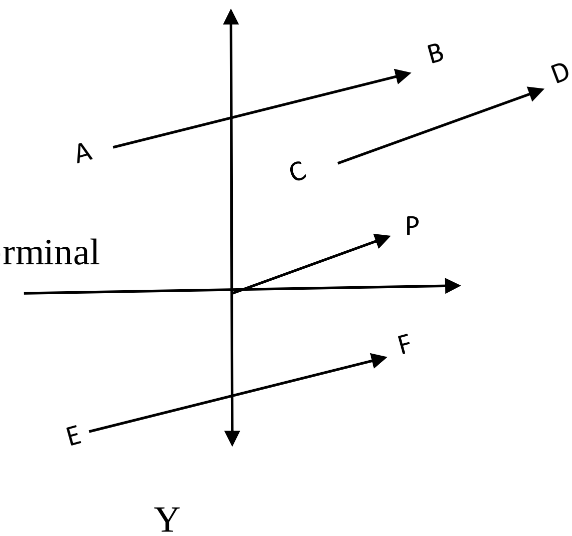
Solution:

Vectors

Definition: Vector. Initial and Terminal point, Length

A vector in the plane is a directed line segment .

The directed line segment $\overrightarrow{AB}$ has initial point A and terminal point B, its length is denoted by $|\overrightarrow{AB}|$. Two vectors are equal if they have the same length and direction.



Definition : (Component Form)

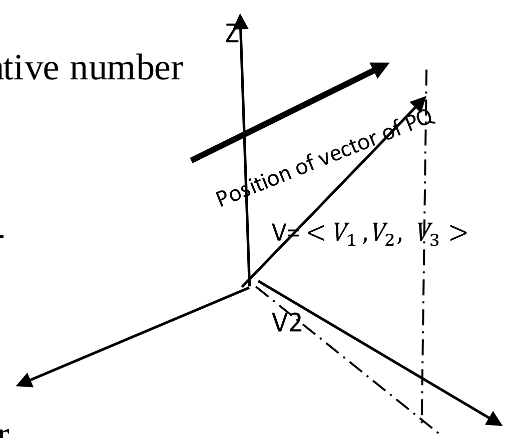
If V is a two –dimensional vector in the plane equal to the vector with the initial point at origin and terminal point(v_1, v_2),Then the component form of v is $V = < v_1, v_2 >$.

If v is a three dimensional vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the component form of v is $V = < v_1, v_2, v_3 >$.

The magnitude or length of the vector V=$\overrightarrow{PQ}$ is the non negative number

$$|V| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z)^2}$$



The only vector with length 0 is the zero vectors

$0 =< 0,0 >$ or $0 =< 0,0,0 >$ this vector is also only vector with no specific direction.

Example: Find (a) component form and (b) length of the vector with initial point $p(-3,4,1)$ and terminal point $Q(-5,2,2)$

Solution:

Vector Algebra Operations

Definitions: (Vector addition and multiplication of vector by a scalar)

Let $u =< u_1 ,u_2,\ u_3 >$ and $v =< v_1 ,v_2,v_3 >$ be vector with k a scalar.

Addition: $u + v\ \ =< u_1 + v_1 ,u_2 + v_2,u_3 + v_3 >$

Scalar multiplication: $ku =< ku_1 ,ku_2,\ ku_3 >$

Example: let $u =< -1,3,1 > and\ v =< 4,7,\ 0 >$. Find

$(a)\ 2u + 3v$ $\qquad (b)\ u - v$ $\qquad (c) \left|\frac{1}{2}\ u\right|$

Properties of vector operations

Let u, v, w be vectors and a, b be scalars

1) $u + v = v + u$ 2) $(u + v) + w = u + (v + w)$ 3) $u + 0 = u$

4) $u + (-u) = 0$ 5) $0\,u = 0$ 6) $1\,u = u$

7) $a(bu) = (ab)u$ 8) $a(u + v) = au + av$ 9) $(a + b)u = au + bu$

Proof 1:

Proof 2:

Proof 3:

Proof 4:

Proof 5:

Proof 6:

Proof 7:

Proof 8:

Proof 9:

Definition: (Unit vectors)

A vector V of length 1 is called a unit vector , the standard unit vectors are :

$i =< 1,0,0 >$, $j =< 0,1,0 >$ and $k =< 0,0,1 >$any vector $v =< v_1 , v_2 , v_3 >$ can be written as a linear combination of standard unit vectors as follows:

$$v =< v_1 , v_2 , v_3 >$$
$$= < v_1 , 0, 0 > +< 0, v_2 , 0 > +< 0,0, v_3 >$$
$$= v_1 < 1,0,0 > +v_2 < 0,1,0 > + v_3 < 0,0,1 >$$
$$= v_1 i + v_2 j + v_3 k$$

Whenever $v \neq 0$, its length $|v|$ is not zero and

$$\left|\frac{1}{|v|} v\right| = \frac{1}{|v|} |v| = 1$$

That is $\frac{v}{|v|}$ is a unit vector in the direction of , called the direction of non zero vector v.

Example: Find a unit vector u in the direction of the vector form $P_1(1,0,0)$ to $P_2(3,2,0)$

Solution:

Definition: (Midpoint)

the midpoint M of the line segment joining points $P_1(x_1, y_1, z_1)$and $P_2(x_2, y_2, z_2)$ in the point $\left(\frac{x_1+x_2}{2} + \frac{y_1+y_2}{2} + \frac{z_1+z_2}{2}\right)$

Example: The midpoint of the segment joining $P_1(3, -2,0)$ $P_2(7,4,4)$ is

Definition: (Dot product)

the dot product $u.v$ (" u dot v") of vectors $u =< u_{1,}u_2 , u_3 >$ and

$v =< v_1, v_2 , v_3 >$ is $uv = u_1 v_1 + u_2 v_2 + u_3 v_3$.

Example: Finding dot products of vectors

a) $< 1, -2, -1 > . < -6,2, -3 > =$

b) $\left(\frac{1}{2}i + 3j + k\right) . (4i - j + 2k) =$

Theorem :(Angle between Two Vectors)

The angle θ between two nan zero vectors $u =< u_1, u_2, u_3 >$ and $v =< v_1, v_2, v_3 >$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|u||v|}\right)$$

Proof: Appling the law of cosines to the triangle in Fig 1, we find that

$$|w|^2 = |u|^2 + |v|^2 - 2|u||v|\cos\theta$$

$$2|u||v|\cos\theta = |u|^2 + |v|^2 - |w|^2$$

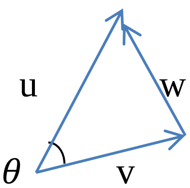


Fig 1

Because $w = u - v$, the component form of w is $< u_1 - v_1, u_2 - v_2, u_3 - v_3 >$. So

$$|u|^2 = \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = u_1^2 + u_2^2 + u_3^2$$

$$|v|^2 = \left(\sqrt{v_1^2 + v_2^2 + u_3^2}\right)^2 = v_1^2 + v_2^2 + v_3^2$$

$$|w|^2 = \left(\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}\right)^2$$

$$= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$

$$= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2$$

and $|u|^2 + |v|^2 - |w|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3)$

Therefore $2|u||v|\cos\theta = |u|^2 + |v|^2 - |w|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3)$

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|u||v|}$$

$$\theta = \cos^{-1}\frac{u_1v_1 + u_2v_2 + u_3v_3}{|u||v|} = \cos^{-1}\left(\frac{u.v}{|u||v|}\right)$$

Example: Find the angle between $u = i - 2j - 2k$ and $v = 6i + 3j + 2k$

Solution:

Definition: (perpendicular (orthogonal) vectors)

Vectors u and v are orthogonal (or perpendicular) if and only if $u.v = 0$.

Example: Appling the definition of orthogonality

(a) $u =< 3, -2 > \, and \, v =< 4,6 >$ are orthogonal because $u.v =$

(b) $u = 3i - 2j + k$ and $v = 2j + 4k$ are orthogonal because $u.v =$

Properties of the Dot Product

If $u.v$ and w are any vectors and c is a scalar, then

1. $u.v = v.u$
2. $(cu).v = u.(cv) = c(u.v)$
3. $u.(v + w) = u.v + u.w$
4. $u.u = |u|^2$
5. $0.u = 0$

Proof 1:

Proof 2:

Proof 3:

Proof 4:

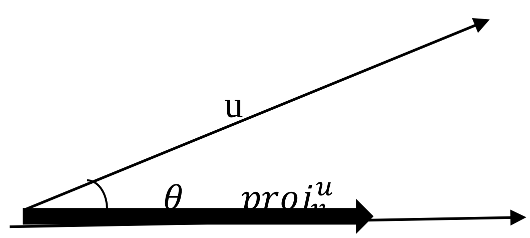
Proof 5:

Vector projection of u onto v:

$$proj_v^u = \left(\frac{u.v}{|v|^2}\right)v$$

Scalar component of u in the direction of v:

$$|u|\cos\theta = \frac{u.v}{|v|} = u.\frac{v}{|v|}$$



Example: Find the vector projection of $u = 6i + 3j + 2k$ on to $v = i - 2j - 2k$ and the scalar component of u in the direction of v.

Solution:

Definition: (Cross product, Vector Product)

$$u \times v = (|u||v| \sin \theta) n$$

The cross product is a vector. For this reason it's also called the vector product of u and v.

Definition: (parallel vector)

Nonzero vectors u and v are parallel if and only if $u \times v = 0$.

Properties of the cross product:

if u, v and w are any vectors and r, s are scalars, then

1. $(ru) \times (sv) = (rs)(u \times v)$
2. $v \times u = -(u \times v)$
3. $u \times (v + w) = u \times v + u \times w$
4. $(v + w) \times u = v \times u + w \times u$
5. $0 \times u = 0$.

Calculating Cross Products Using Determinates:

If $u = u_1 i + u_2 j + u_3 k$ *and* $v = v_1 i + v_2 j + v_3 k$, then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example: Find $u \times v$ and $v \times u$ if $u = 2i + j + k$ and $v = -4i + 3j + k$

Solution:

Calculating the triple scalar product

$$(u \times v).w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example: Find the volume of the box (parallelepiped) determined by

$u = i + 2j + k$, $v = -2i + 3k$ and $w = 7j - 4k$.

Solution:

Lines and Planes in Space

Suppose that L is a line in space passing through a point $p_o(x_o, y_o, z_o)$ parallel to a vector $v = v_1 i + v_2 j + v_3 k$. Then L is the set of all points $p(x, y, z)$for which $\overrightarrow{p_o p}$ is parallel to v, thus $\overrightarrow{p_o p} = tv$ for some scalar parameter t.

$$(x - x_o)i + (y - y_o)j + (z - z_o)k = t(v_1 i + v_2 j + v_3 k)$$

Which can be rewritten as

$$xi + yj + zk = x_o i + y_o j + z_o k + t(v_1 i + v_2 j + v_3 k)$$

$$x = x_o + tv_1 \quad , \qquad y = y_o + tv_2 \ , \qquad z = z_o + tv_3$$

Parametric Equation for a line

The standard parameterization of the line through $p_o(x_o, y_o, z_o)$ parallel to $v = v_1 i + v_2 j + v_3 k$ is $x = x_o + tv_1 \, , y = y_o + tv_2 \, , z = z_o + tv_3 \, , -\infty < t < \infty$

Example: Find parametric equation for the line through $(-2,0,4)$ parallel to $v = 2i + 4j - 2k$.

Solution:

Example: Parameterizing a line through two points, Find parametric equation for the line through $p(-3,2,-3)$ and $Q(1,-1,4)$.

Solution:

An equation for a Plane in Space

Suppose that plane M passes through a point $p_o(x_o, y_o, z_o)$ and is normal to the nonzero vector $n = Ai + Bj + Ck$. Then M is the set of all points $p(x, y, z)$ for which $\overrightarrow{p_o p}$ is orthogonal to n

$$n.\overrightarrow{p_o p} = 0$$

$$(Ai + Bj + Ck).[(x - x_o)i + (y - y_o)j + (z - z_o)k] = 0$$

$$A(x - x_o) + B(y - y_o) + C(z - z_o) = 0$$

Equation for a Plane:

The plane through $P_o(x_o, y_o, z_o)$ normal to $n = Ai + Bj + Ck$

Vector equation: $n.\overrightarrow{p_o p} = 0$

Component equation: $A(x - x_o) + B(y - y_o) + C(z - z_o) = 0$

Component equation simplified: $Ax + By + Cz = D$, where

$$D = Ax_o + By_o + Cz_o.$$

Example: Find an equation for the plane through $p_0(-3,0,7)$ perpendicular to $n = 5i + 2j - k$.

Solution:

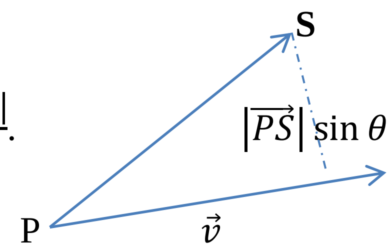
Example: Find an equation for the plane normal to $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.

Solution:

Example: Find the point where the line $x = \frac{8}{3} + 2t,\quad y = -2t,\quad z = 1 + t$ intersect the plane $3x + 2y + 6z = 6.$

Solution:

Definition: The Distance from a Point to a Line in Space



Distance from a point S to a line through P parallel to v is $d = \frac{|\overrightarrow{PS}\times v|}{|v|}$.

Example: Find the distance from a point $S(1,1,5)$ to the line

$$L: x = 1 + t,\ \ y = 3 - t, z = 2t$$

Solution:

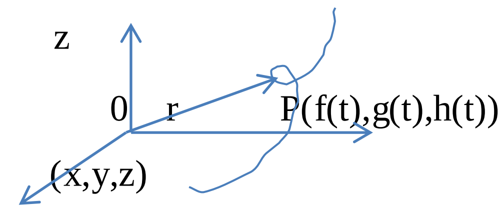
Vector Valued Functions in Space

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), \quad y = g(t), \quad z = h(t) \quad t \in I \text{ ....(1)}$$

The points $(x, y, z) = \big(f(t)g(t), h(t)\big)$, $t \in I$ make up the **curve** in space that we call the particle's **path**. The equations and interval in Equation (1) **parameterize** the curve. A curve in space can also be represented in vector form. The vector

$$r(t) = \overrightarrow{OP} = f(t)i + g(t)j + h(t)k \text{ ....(2)}$$



from the origin to the particle's **position** $P(\mathrm{f}(t), g(t), h(t))$ at time t is the particle's **position. vector** (Fig 2). The functions $\mathrm{f}, g, and\ h$ are the **component functions (components)** of the position vector. We think of the particle's path as the **curve traced by *r*** during the time interval I. Displays several space curves generated by a computer graphing program. It would not be easy to plot these curves by hand.

Equation (2) defines ***r*** as a vector function of the real variable t on the interval I. More generally, a **vector function** or **vector-valued function** on a domain set D is a rule that assigns a vector in space to each element in D. For now, the domains will be intervals of real numbers resulting in a space curve. Later, in Chapter 16, the domains will be regions.

Example: Find the value of a component function at $t = 0$ for the following vector valued functions

1. $r(t) = \cos t\, i + \sin t\, j + tk$
2. $r(t) = t^2 i + \sin t\, j + tk$.

Solution:

Limits and Continuity

The way we define limits of vector-valued functions is similar to the way we define limits of real-valued functions.

Definition: (Limit of Vector Functions)

Let $r(t) = f(t)i + g(t)j + h(t)k$be a vector function and $\boldsymbol{L}$ a vector. We say that

$\boldsymbol{r}$ has **limit L** as t approaches t and write

$$\lim_{t \to t_0} r(t) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all t

$$0 < |t - t_0| < \delta \quad \Longrightarrow \quad |r(t) - L| < \varepsilon.$$

Example: If $r(t) = \cos t\, i + \sin t + tk$, then find the limit of r at $t \to \frac{\pi}{4}$.

Solution:

We define continuity for vector functions the some way we define continuity for scalar function.

Definition: (Continuous at a Point)

A vector function $\boldsymbol{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} r(t) = r(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

Examples: (a) All the following functions are continuous at every value of t in $(-\infty, \infty)$, because their component functions are continuous at every value of t in $(-\infty, \infty)$.

1. $r(t) = (\cos t)i + (\sin t)j + tk$
2. $r(t) = (\cos t)i + (\sin t)j + 0.3tk$
3. $r(t) = (\cos 3t)(\sin t)i + (\sin 3t)j + t^2 k$
4. $r(t) = (\cos t)i + (\sin t)j + (\sin 2t)k$

(b) The function $r(t) = (\cos t)i + (\sin t)j + \lfloor t \rfloor k$

is discontinuous at every integer, where the greatest integer function $\lfloor t \rfloor$is discontinuous.

Definition: (Derivative)

The vector function $r(t) = f(t)i + g(t)j + h(t)k$ has a **derivative (is differentiable)** at t if f, g, and h have derivatives at t. The derivative is the vector function

$$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t} = \frac{df}{dt}i + \frac{dg}{dt}j + \frac{dh}{dt}k.$$

Example: Derivative the function

$$r(t) = (3\cos t)i + (3\sin t)j + t^2 k.$$

Solution:

Differentiation Rules for Vector Functions

Let u and v be differentiable vector functions of t, C a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}[C] = 0$

2. *Scalar Multiple Rules:* $\frac{d}{dt}[cu(t)] = cu'(t)$

$$\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$$

3. *Sum Rule:* $\frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$

4. *Difference Rule:* $\frac{d}{dt}[u(t) - v(t)] = u'(t) - v'(t)$

5. *Dot Product Rule:* $\frac{d}{dt}[u(t). v(t)] = u'(t)v(t) + u(t)v'(t)$

6. *Cross Product Rule:* $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$

7. *Chain Rule:* $\frac{d}{dt}[u(f(t))] = f'(t)u'(t).$

Proof of the Dot Product Rule:

Suppose that $u(t) = u_1(t)i + u_2(t)j + u_3(t)k$ and $v(t) = v_1(t)i + v_2(t)j + v_3(t)k$,

Then $\frac{d}{dt}(u.v) = \frac{d}{dt}(u_1.v_1 + u_2.v_2 + u_3.v_3)$

$$= (u_1'.v_1 + u_2'.v_2 + u_3'.v_3) + (u_1.v_1' + u_2.v_2' + u_3.v_3')$$

$$= u'v + uv'$$

Integral of Vector Functions

Definition: (Defined Integral)

If the components of $r(t) = f(t)i + g(t)j + h(t)k$ are integrable over $[a, b]$, then so is r, and the definite integral of r from a to b is

$$\int_a^b r(t)dt = \left(\int_a^b f(t)dt\right)i + \left(\int_a^b g(t)dt\right)j + \left(\int_a^b h(t)dt\right)k.$$

Example: Evaluating Definite Integral

$$\int_0^{\pi}[(cost)i + j - 2tk]dt =$$

Solution:

Arc Length along a Space Curve

Definition: (Length of a Smooth Curve)

The Length of a Smooth Curve $r(t) = x(t)i + y(t)j + z(t)k, a \leq t \leq b$

That traced exactly ones as t increases from $t = a$ to $t = b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example: Find the Length of the curve $r(t) = \cos ti + \sin tj + tk \ for \ t \in [0, 2\pi]$.

Solution:

Directional Derivatives and Gradient Vector

If $f(x, y)$ is differentiable, then the **rate** at which f changes with respect to t along a differentiable curve $x = g(t),\ y = h(t)$ is $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

At any point $p_0(x_0, y_0) = p_0\ (g\ (t_0), h(t_0))$, this equation gives the rate of change of f with respect to increasing t and therefore depends, among other things on the direction of motion along the curve.

Suppose that the function $f(x, y)$ is defined throughout a region R in the $xy-\ plane$, that $p_0(x_0, y_0)$ is a point in R, and that $u = u_1\ i\ +\ u_2\ j$ is a unit vector. Then the equation $x = x_0 + su_1\ ,\ y = y_0 + su_2$

Parameterize the line through p_0 parallel to u.

Definition: (Directional Derivative)

The Derivative of f at $p_{q0}(x_0, y_0)$ in the direction of the unit vector $u = u_1\ i\ +\ u_2\ j$ is the number $(\frac{df}{ds})_{u,p_0} = lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$ provided the limit exists.

The directional derivative is also denoted by

$(D_u f)_{p_0}$ the derivative of f at p_0 in the direction of u.

Example: Finding a Directional Derivative using the Definition

Find the Derivative of $f(x, y) = x^2\ +\ xy$ at $p_0(1,2)$ in the direction of the unit vector

$u = (\frac{1}{\sqrt{2}})i + (\frac{1}{\sqrt{2}})j.$

Solution:

Definition: (Gradient Vector)

The **gradient vector (gradient)** of $f(x,y)$ at a point $p_0(x_0,y_0)$ is the vector

$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$ obtained by evaluating the partial derivatives of f at p_0 .

Example: Find the gradient of $f(x,y) = x^2 + xy$ at a point $p_0(1,2)$.

Solution:

Theorem: (The Directional Derivative is a Dot Product)

If $f(x,y)$ is differentiable in an open region containing $p_0(x_0,y_0)$, then

$$\left(\frac{df}{ds}\right)_{u,p_0} = (\nabla f)_{p_0} . u$$

The dot product of the gradient f at p_0 and u.

Example: find the directional derivative of $f(x,y) = x^2 + xy$ at $p_0(1,2)$ and $u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$.

Solution:

Properties of the Directional Derivative $D_u f$

$\nabla f.\mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when u is the direction of ∇f that is, at each point p in it is domain, f increases most rapidly in the direction of the gradient vector ∇f at p. The derivative in this direction is
$$D_u f = |\nabla f| \cos\theta = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction $-\nabla f$. The derivative in this direction is
$$D_u f = |\nabla f| \cos\pi = -|\nabla f|.$$
3. Any direction u orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ equals $\frac{\pi}{2}$ and $D_u f = |\nabla f| \cos\pi = 0.|\nabla f| = 0$.

as we discuss later, these properties hold in three dimensions as well as two.

Example: Find the direction derivative in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$

a) Increases most rapidly at the point $(1, 1)$.

b) Decreases most rapidly at the point $(1, 1)$.

c) What are the directions of zero change in f at the point (1,1)?

Solution:

Algebra Rules for Gradients:

1. Constant Multiple Rule: $\nabla(kf) = k\nabla f$ (any number k).
2. Sum Rule: $\nabla(f+g) = \nabla f + \nabla g$
3. Difference Rule: $\nabla(f-g) = \nabla f - \nabla g$
4. Product Rule: $\nabla(f.g) = f\nabla g + g\nabla f$
5. Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$.

Example: We illustrate the rules with $f(x,y) = x - y$, $g(x,y) = 3y$

Solution:

Tangent plans and Differentials

Tangent plans and Normal lines

If $r = g(t)i + h(t)j + k(t)k$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable function f, then $f\big(g(t),h(t),k(t)\big) = c$. Differentiating both sides of this equation with respect to t leads to

$$\frac{d}{dt} f\big(g(t),h(t),k(t)\big) = \frac{d}{dt}c$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0 \qquad \text{(Chain Rule)}$$

at every point along the curve. ∇f is orthogonal to the curve's velocity vector.

Definition: (Tangent Plane, Normal Line)

The tangent plane at the point $p_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through p_0 normal to $\nabla f\big|_{p_0}$.

The normal line of the surface at p_0 is the line through p_0 parallel to $\nabla f\big|_{p_0}$.

Tangent plane to $f(x, y, z) = c$ at $p_0(x_0, y_0, z_0)$

$$f_x(p_0)(x - x_0) + f_y(p_0)(y - y_0) + f_z(p_0)(z - z_0) = 0.$$

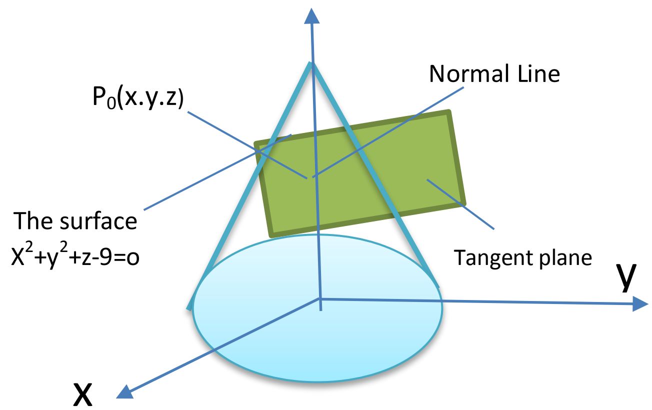
Normal Line to $f(x, y, z) = c$ at $p_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(p_0)t, \quad y = y_0 + f_y(p_0)t, \quad z = z_0 + f_z(p_0)t.$$

Example: Find the tangent plane and normal line of the surface

$f(x, y, z) = x^2 + y^2 + z - 9 = 0$ (a circular paraboloid) at the point $p_0(1,2,4)$.

Solution:



Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $p_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$.

Example: Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0,0,0)$.

Solution:

Example: Tangent line to the curve of intersection of two surfaces. The surfaces $f(x, y, z) = x^2 + y^2 - 2 = 0$ a cylinder and $g(x, y, z) = x + z - 4 = 0$ a plane. Finding parametric equations for the line tangent to E at the point $p_0(1,1,3)$.

Solution:

Definition: Total Differential

If we move from (x_0, y_0) to the point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

In the linearization of *f* is called the total differential of *f*.

Example: Find the total differential of $f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the point $(3,2)$.

Solution:

Extreme Values and Saddle Points

Derivative Tests for Local Extreme Values

Definition: (Local Maximum, Local Minimum)

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$, for all domain points (x, y) in an open disk centered at (a, b).

Theorem: (First derivative test for local extreme value)

If $f(x, y)$ has a local maximum or local minimum value at an interior point (a, b) of it is domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Definition: (Critical point)

An interior point (a, b) of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both f_x and f_y do not exist is a critical point of f.

Example: Find the critical point of $f(x, y) = \sqrt[3]{x + y}$

Solution:

Definition: (Saddle point)

A differentiable function $f(x, y)$has a Saddle point at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $\left(a, b, f(a, b)\right)$ on the surface $z = f(x, y)$ is called Saddle point of the surface.

Example: Find the local extreme value of $f(x, y) = x^2 + y^2$.

Solution:

Theorem: Second Derivative Test for Local Extreme Values

Suppose that $f(x,y)$ and its first and second derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Then

i. f has a local maximum at (a,b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at (a,b).

ii. f has a local minimum at at (a,b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at (a,b).

iii. f has a saddle point at (a,b) if and $f_{xx}f_{yy} - {f_{xy}}^2 < 0$ at (a,b).

iv. The test is inconclusive at at (a,b) if and $f_{xx}f_{yy} - {f_{xy}}^2 = 0$ at (a,b).

In this case, we must find some other way to determine the behavior of f at (a,b).

The expression $f_{xx}f_{yy} - {f_{xy}}^2$ is celled the discriminant hessian or Hessian of f. it is sometimes easier to remember in determinant from, $f_{xx}f_{yy} - {f_{xy}}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$.

Example: Find the local extreme value of the function

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution:

Example: Find local extreme values of $f(x,y) = xy$.

Solution:

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x,y)$ on a closed and bounded region R in to three steps

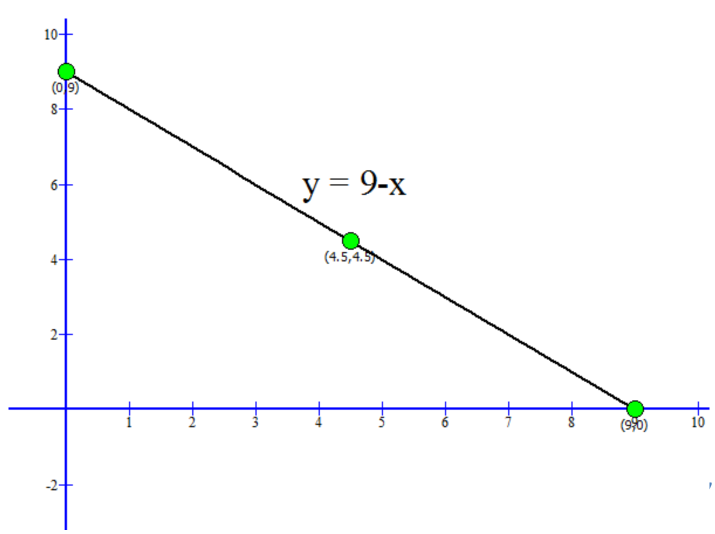
1. List the interior points of R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.
2. List the boundary points of R where f has a local maxima and minima and evaluate f at these points. We show to do this shortly.
3. Look through the lists for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R. Since absolute maxima

and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists in Step1 and 2.

Example: Find the absolute maximum and minimum values of

$f(x,y) = 2 + 2x + 2y - x^2 - y^2$.

Solution:



Lagrange Multipliers

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable. To find the local maximum and minimum values of subject to the constraint $g(x, y, z) = 0$, find the values of x, y, z and λ that simultaneously satisfy the equations $\nabla f = \lambda \nabla g$ and $g(x, y, z) = 0$

For functions of two independent variables, the condition is similar, but without the variable z.

Example: Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipes $\frac{x^2}{8} + \frac{y^2}{2} = 1.$

Solution:

Example: Find the maximum and minimum values of the function

$f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution: