Salahaddin University-Erbil	**Subject: Analytical Mechanics**
College of Science	**Period: 2 hours**
Department of Physics	**Date: January 2021**
2nd Stage/General	**Final Examination**
First Semester 2021-2022	**First Trial**

Q.1/ Find the transformation matrix for a rotation of the coordinate axes about z-axis through an angle φ followed by a rotation about y'-axis through an angle .

[15 Marks]

Q.2/ For harmonic oscillator system with: $F(x) = -kx$. What are?

(i) The potential energy, total energy and kinetic energy for a system.
(ii) Plot the graph of the energies of (i) as a functions of time t and position x.
(iii) Turning points of motion. *[15 Marks]*

Q.3/ For vertical motion with linear drag:

(I) What is the physical differential equation of motion?
(II) Define the terminal speed and characteristic time.
(III) At what time the speed of particle v_y is at 95% of v_{ter}?
(IV) Plot v_y versus t for $v_{yo} > v_{ter}$? *[15 Marks]*

Q.4/

(a) Show that: $\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$ *[5+5+5 Marks]*

Using physical representation of the unit vectors in polar coordinates.

(b) Show that the tangential component of acceleration is given by $a_\tau = \frac{\vec{v}.\vec{a}}{v}$

(c) Find the corresponding point of (1,1,1) in Cylindrical Coordinates.

Good Luck *Asst. Prof. Dr. Tahseen G. Abdullah*

Ans. of Q.1:

1.10: Verify the transformation matrix for a rotation of the coordinate axes about z-axis through an angle ϕ followed by a rotation about y'-axis through an angle θ .

About new y' axis | About z axis

$$\cdots \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow 1^{st} \text{ change}$$

2^{nd} change

Successive changes

$$\Rightarrow \cdots \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

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Ans. of Q.2:

Energy Consideration in SHM

$x = x_m \cos(\omega t + \phi)$ | $\dot{x} = -x_m \omega \sin(\omega t + \phi)$

1.The potential energy

$$V = -\int F(x)dx = -\int -kx dx = \frac{1}{2}kx^2$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$

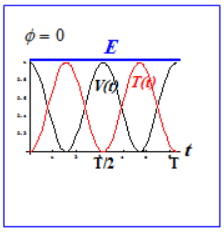
2.The kinetic energy

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$

3. The total energy is

$$E = T + V = \frac{1}{2}kx_m^2 = \frac{1}{2}kA^2 = const.$$



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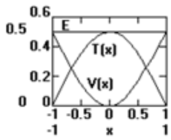
$$F(x) = -\frac{dV(x)}{dx} = -kx$$

$$V(x) = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

$$E = T_\circ = \frac{1}{2}kA^2$$

$$T_\circ = T(x) + V(x)$$

$$T(x) = T_\circ - V(x) = \frac{1}{2}k(A - x^2)$$



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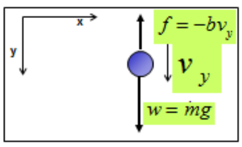
$$turning\ points\ @\ T(x_1) \to 0 \quad \therefore x_1 = \pm A$$

Ans. of Q.3:

Vertical Motion with Linear Drag

- Consider motion of an object thrown vertically downward and subject to gravity and linear air resistance.
- The equation of vertical motion for linear drag is determined by:

$$m\dot{v}_y = mg - bv_y$$



- Gravity accelerates the object down, the speed increases until the point when the retardation force becomes equal in magnitude to gravity. One then has *terminal speed.*

$$0 = mg - bv_y \Rightarrow v_{ter} = v_y(a = 0) = \frac{mg}{b}$$

Note: v_{ter} depends on mass m and linear drag coefficient b.
i.e terminal speed is different for different objects.

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- We found $v_y = v_{y0}e^{-t/\tau} + v_{ter}\left(1 - e^{-t/\tau}\right)$
- At $t=0$, one has $v_y = v_{y0}$
- Whereas for $t \to \infty$ $v_y = v_{ter}$
- As the simplest case, consider $v_{y0}=0$, i.e. dropping an object from rest.

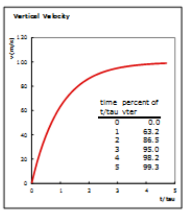
$$v_y = v_{ter}\left(1 - e^{-t/\tau}\right)$$

- After an interval $t = 1\tau$

$$v_y = v_{ter}(1 - e^{-1}) = 0.63 v_{ter}$$

- By the time $t = 3\tau$ the projectile velocity is at 95% of v_{ter}.
- After an interval $t = 10\tau$

$$v_y = 0.99995\, v_{ter} \approx v_{ter}$$



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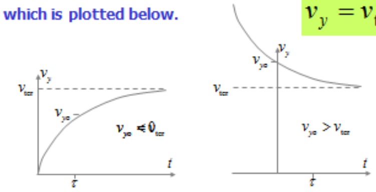
Plot of v_y versus t for different v_{y0}:

- Let's take a look at the solution for $v_{y0}=0$ (dropping the projectile from rest). In this case, the equation is just

$$v_y = v_{ter}(1-e^{-t/\tau})$$

which is plotted below.



- Note that it is not enough to simply derive an equation. To really understand the motion you need to sketch such plots, or look at limiting behavior (e.g. position and velocity as $t\to\infty$).

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- One can write equation of vertical motion as: $m\dot{v}_y = mg - bv_y$

$$m\dot{v}_y = -b\left(v_y - v_{term}\right)$$

- Or in terms of differentials $m dv_y = -b\left(v_y - v_{term}\right)dt$
- Separate variables then change variable:

$$\frac{dv_y}{v_y - v_{term}} = -\frac{bdt}{m} \qquad u = v_y - v_{term}, \quad du = dv_y$$

- So we have $\frac{du}{u} = -\frac{bdt}{m} = -kdt \longrightarrow \int \frac{du}{u} = -k\int dt \longrightarrow \ln u = -kt + C$
- Or...

$$u = Ae^{-kt} \longrightarrow v_y - v_{ter} = Ae^{-t/\tau} \longrightarrow \tau = 1/k = m/b$$

is the Characteristic time

- Now apply initial conditions: when $t=0, v_y = v_{y0}$ $\longrightarrow A = v_{y0} - v_{ter}$
- The velocity as a function of time is thus given by

$$v_y = v_{ter} + \left(v_{y0} - v_{ter}\right)e^{-t/\tau}$$

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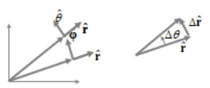
Ans. of Q.4:

Derivatives of Polar Unit Vectors: $\frac{d\hat{\mathbf{r}}}{dt}$

- As the coordinate r changes from time t_1 to time $t_2 = t_1 + \Delta t$, the unit vector $\hat{\mathbf{r}}$ changes by:



Recall: arc length $= r\theta$
In this case, $\theta = \Delta\theta$ and $r = |\hat{\mathbf{r}}| = 1$

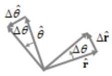
$|\Delta\hat{\mathbf{r}}| = \Delta\theta$

$\Delta\hat{\mathbf{r}} = \Delta\theta\,\hat{\theta}$

- after taking the limit as Δt approaches zero,

$$\lim_{\Delta t\to 0}\frac{\Delta\hat{\mathbf{r}}}{\Delta t} = \lim_{\Delta t\to 0}\left(\frac{\Delta\theta}{\Delta t}\hat{\theta}\right)$$

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\hat{\theta}$$



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p.16: Show that the tangential component of acceleration is given by:

$$a_\tau = \frac{\bar{v}.\bar{a}}{v} \qquad a_n = (a^2 - a_\tau^2) = (a^2 - \frac{(\bar{v}.\bar{a})^2}{v^2})$$

and the normal component is therefore:

$$\bar{v} = v\hat{\tau} \text{ and } \bar{a} = a_\tau\hat{\tau} + a_n\hat{n}$$

$$\bar{v}\cdot\bar{a} = va_\tau, \text{ so } a_\tau = \frac{\bar{v}\cdot\bar{a}}{v}$$

$$a^2 = a_\tau^2 + a_n^2, \text{ so } a_n = \left(a^2 - a_\tau^2\right)^{\frac{1}{2}}$$

$$\bar{v}\cdot\bar{v} = v^2$$

$$\frac{d\bar{v}}{dt}\cdot\bar{v} + \bar{v}\cdot\frac{d\bar{v}}{dt} = 2v\dot{v}$$

$$2\bar{v}\cdot\bar{a} = 2v\dot{v}$$

$$\bar{v}\cdot\bar{a} = v\dot{v}$$

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Ex.: What is the corresponding point of (1,1,1) in Cylindrical Coordinates.

Sol.:

$$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(1) = 45$$

$$z = z = 1$$

Thus, the corresponding point in Cylindrical Coordinates is: (√2,45,1)

Ex.: What is the corresponding point of (√2,45°,1) in Cartesian Coordinates. (H.W.)

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