Salahaddin University-Erbil College of Science Department of Physics 2^{nd} Stage/General First Semester 2021-2022	Subject: Analytical Mechanics Period: 2 hours Date: June 2021 Final Examination Second Trial

Q.1/ Express the vector $\vec{A} = 4\hat{\imath} + \hat{\jmath} - 2\hat{k}$ in terms of the triad $\hat{\imath}', \hat{\jmath}', \hat{k}'$ for rotation of the coordinate system about the **y-axis** through an angle $30°$. Show that the magnitude of the vector $\vec{A}$ is unchanged after the rotation. *[15 Marks]*

Q.2/ *[5+5+5 Marks]*

(a) What is the relative motion?

(b) What is the corresponding point of **(1,1,1)** in spherical coordinates?

(c) For a particle moving in a circular path of radius ***2m*** with a velocity function $v=2t^2 m/s$. What is the magnitude of its total acceleration at $t = 1\ s$ in ***n-t*** coordinates?

Q.3/ For harmonic oscillator system with: $F(x) = -kx$: *[15 Marks]*

(i) What is the solution of the system?

(ii) Plot the graph of the position, velocity and acceleration as a function of time t.

(iii) What are the potential energy, total energy and kinetic energy for a system?

(iv) Plot the graph of the energies of (iii) as a functions of position x.

Q.4/ *[5+5+5 Marks]*

(A) A block is projected with initial velocity v_o on a smooth horizontal plane, but that there is air resistance proportional to v: $F(v) = -cv$, where c is constant. Find the speed as a function of position $v(x)$?

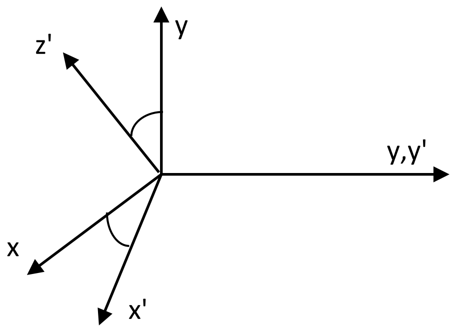
(B) What is the mechanical energy in damping harmonic oscillator?

(C) For variation of gravity with height: What is the magnitude of the free-fall acceleration at a point that is a distance $2R_e$ above the surface of the Earth, where R_e is the radius of the Earth?

Good Luck *Asst. Prof. Dr. Tahseen G. Abdullah*

Ans. of Q.1:

$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} \hat{i}.\hat{i}' & \hat{j}.\hat{i}' & \hat{k}.\hat{i}' \\ \hat{i}.\hat{j}' & \hat{j}.\hat{j}' & \hat{k}.\hat{j}' \\ \hat{i}.\hat{k}' & \hat{j}.\hat{k}' & \hat{k}.\hat{k}' \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$



$$= \begin{bmatrix} \cos 30 & \cos\frac{\pi}{2} & \cos(\frac{\pi}{2}+30) \\ \cos\frac{\pi}{2} & \cos 0 & \cos\frac{\pi}{2} \\ \cos(\frac{\pi}{2}-30) & \cos\frac{\pi}{2} & \cos 30 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3}+1 \\ 1 \\ 2-\sqrt{3} \end{bmatrix} = \begin{bmatrix} 4.464 \\ 1 \\ 0.268 \end{bmatrix} \qquad \begin{matrix} A_{x'} = 4.464 \\ A_{y'} = 1 \\ A_{z'} = 0.268 \end{matrix}$$

$$\vec{A}' = (2\sqrt{3}+1)\hat{i}' + 4\hat{j}' + (2-\sqrt{3})\hat{k}'$$

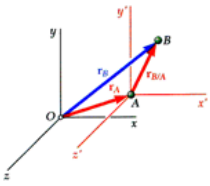
$$A'^2 = (2\sqrt{3}+1)^2 + (1)^2 + (2-\sqrt{3})^2$$
$$= 12 + 1 + 4\sqrt{3} + 1 + 4 - 4\sqrt{3} + 3 = 21$$

$$A^2 = (4)^2 + (1)^2 + (-2)^2 = 16 + 1 + 4 = 21$$

Ans. of Q.2:

(a)

Relative Motion



- Designate one frame as the *fixed frame of reference*. All other frames not rigidly attached to the fixed reference frame are *moving frames of reference*.
- Position vectors for particles A and B with respect to the fixed frame of reference $Oxyz$ are $\vec{r}_A$ and $\vec{r}_B$.
- Vector $\vec{r}_{B/A}$ joining A and B defines the position of B with respect to the moving frame $Ax'y'z'$ and
  $\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$
- Differentiating twice,
  $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ $\quad \vec{v}_{B/A}$ = velocity of B relative to A.
  $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ $\quad \vec{a}_{B/A}$ = acceleration of B relative to A.
- Absolute motion of B can be obtained by combining motion of A with relative motion of B with respect to moving reference frame attached to A.

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(b)

Ex.: What is the corresponding point of (1,1,1) in Spherical Coordinates.

Sol.:

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{1+1}}{1}\right) = \tan^{-1}(\sqrt{2}) = 54.73$$

$$or \quad \theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.73$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = 45$$

Thus, the corresponding point in Spherical Coordinates is: (√3,54.73,45)

Ex.: What is the corresponding point of (√3,54.73°,45°) in Cartesian Coordinates. (H.W.)

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(c)

magnitude of total acceleration

$$\vec{a} = \dot{v}\hat{\tau} + \frac{v^2}{R}\hat{n} \qquad v = 2t^2 \Rightarrow \dot{v} = 4t \Rightarrow at\ t = 1s \quad v = 2\,m/s \ \&\ \dot{v} = 4\,m/s^2$$

$$a = \sqrt{\dot{v}^2 + \frac{v^4}{R^2}} = \sqrt{4^2 + \frac{2^4}{2^2}} = \sqrt{16+4} = \sqrt{20} = 4.47\,m/s^2$$

Ans. of Q.3:

2nd way Let the trial solution of Eq(1) is $x = e^{qt}$ $\qquad \boxed{\ddot{x} + \frac{k}{m}x = 0}$

Then: $\dot{x} = qe^{qt}$ & $\ddot{x} = q^2 e^{qt}$

Substituting into Eq.(1), we obtain:

$$(q^2 + \frac{k}{m})e^{qt} = 0 \rightarrow e^{qt} \neq 0 \Rightarrow \therefore (q^2 + \frac{k}{m}) = 0 \qquad \therefore q = \pm\sqrt{-\frac{k}{m}} = \pm i\omega$$

Thus, there are two solutions of Eq(1): $e^{+i\omega t}$ & $e^{-i\omega t}$

The general solution is a linear combination of them:

$$x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

Or $\quad x = A\cos(\omega t + \phi) \rightarrow \boxed{x = x_m \cos(\omega t + \phi)}$

where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the motion

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$$\boxed{x = x_m \cos(\omega t + \varphi)}$$

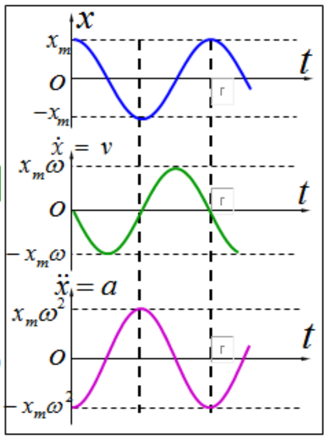
$$T = \frac{2\pi}{\omega} \qquad \varphi = 0$$

$$\boxed{\dot{x} = v = -x_m\omega \sin(\omega t + \varphi)}$$

$$= x_m\omega\cos(\omega t + \varphi + \frac{\pi}{2})$$

$$\boxed{\ddot{x} = a = -x_m\omega^2 \cos(\omega t + \varphi)}$$

$$= x_m\omega^2 \cos(\omega t + \varphi + \pi)$$



Spring & SHM

Energy Consideration in SHM

$\boxed{x = x_m \cos(\omega t + \phi)}$ $\boxed{\dot{x} = -x_m \omega \sin(\omega t + \phi)}$

1.The potential energy

$$V = -\int F(x)dx = -\int -kx dx = \frac{1}{2}kx^2$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$

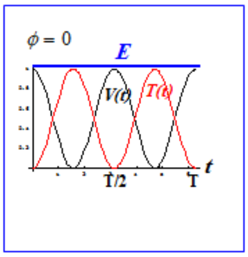
2.The kinetic energy

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$

3. The total energy is

$$E = T + V = \frac{1}{2}kx_m^2 = \frac{1}{2}kA^2 = const.$$



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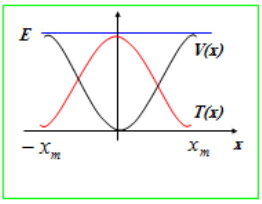
- In the Figure, both potential and kinetic energies oscillate with time t and vary between zero and maximum value of $\frac{1}{2}kx_m^2 \,(or \,\frac{1}{2}kA^2)$
- Both V and T vary with twice the frequency of the displacement and velocity.
- This Figure represents the variation of kinetic and potential energies with displacement in SHM.

$$E = T + V = \frac{1}{2}kx_m^2 = const.$$



$$V(x) = \frac{1}{2}kx^2$$

$$T(x) = E - V(x)$$

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Ans. of Q.4:

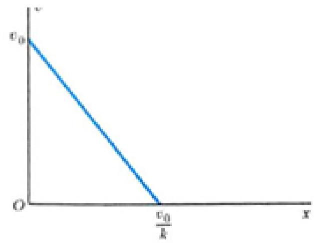
(A)

Determine the v in terms of x.

$$a = \frac{dv}{dx}\frac{dx}{dt} = -kv \Rightarrow v\frac{dv}{dx} = -kv$$

$$\Rightarrow \int_{v_0}^{v} dv = -k\int_0^x dx$$

$$v = v_0 - kx$$



(B)

Ex. What is the Mechanical Energy in DHO?

The total energy of the DHO is: $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \neq const.$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x} = -c\dot{x}^2$$

The rate of change of total energy is the product of the damping force and the velocity. This is always negative and represents the rate at which the energy is being dissipated into heat by friction.

When the damping force is not large, the mechanism energy is:

$$E(t) = \frac{1}{2}kA^2 e^{-2t/\tau}$$

This Eq. shows that *the mechanical energy of the oscillator decreases exponentially with time.*

The energy decreases twice as rapidly as the amplitude.

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(C)

Consider an object of mass m at a height h above the Earth's surface.

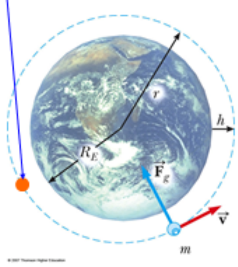
$$F = -G\frac{mM_E}{r^2} = -G\frac{mM_E}{(R_E + h)^2}$$

Acceleration g due to the gravity is:

$$F = -G\frac{mM_E}{(R_E + h)^2} = -mg$$

g will vary with altitude (height):

$$g = G\frac{M_E}{(R_E + h)^2}$$



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