Salahaddin University-Erbil
College of Science
Department of Mathematics
3rd Stage

Subject: Analytical Mechanics
Period: 60 Minutes
Date: 21 November 2021
First Examination
First Semester 2021-2022

Q.1/ For rotation of the coordinate system through an angle θ: *[9+9 Marks]*

i) Show that the components of a vector $\vec{V}$ in two dimensions are given by:

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

ii) What are the properties of the above transformation (rotation) matrix?

Q.2/ Show that: $\dfrac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{\mathbf{r}}$ *[7 Marks]*
Using physical representation of the unit vectors in polar coordinates.

Q.3/ Choose the ***correct*** answer: *[10 Marks]*

a) Consider a particle moving in a straight line with $x = 6t^2 - t^3$ *(meters)*, the maximum velocity occurs at the time: *(0 seconds , 2 seconds , 4 seconds, 6 seconds , None of them)*

b) Given the velocity of a particle in rectilinear motion varies with the displacement x according to the equation: $\dot{x} = v(x) = \frac{2}{x}$. What is the force acting on the particle as a function of x ? *($-2m\,x^{-3}$, $-4m\,x^{-3}$, $-2m\,x^{-2}$, $-4m\,x^{-2}$, None of them)*

c) The acceleration of a particle sliding from rest down an inclined plane θ with coefficient of kinetic friction μ is positive when:
$(\sin\theta > \mu\cos\theta \ , \ \sin\theta < \mu\cos\theta \ , \ \sin\theta = \mu\cos\theta \ , \ \sin\theta \le \mu\cos\theta \ , \ \textit{None of them})$

d) If a particle moves in a circular path (polar coordinates) with constant velocity, its radial acceleration is: $(\textit{Zero}, \ r\ddot{\theta}, \ -r\dot{\theta}^2, \ 2\dot{r}\dot{\theta} \ , \ \textit{None of them})$

e) If the Cartesian point $(x, y, z) = (1, \sqrt{3}, 2)$. The corresponding point in cylindrical coordinates is: $[(2, \frac{\pi}{3}, 2), \ (2, \frac{\pi}{6}, 2), \ (\sqrt{8}, \frac{\pi}{3}, 2), \ (\sqrt{8}, \frac{\pi}{6}, 2), \ \textit{None of them}]$

Good Luck *Asst. Prof. Dr. Tahseen G. Abdullah*

Ans. of Q.1:

a)

▪Generally, let $\hat{e}_1$ and $\hat{e}_2$ are base vectors, i.e.

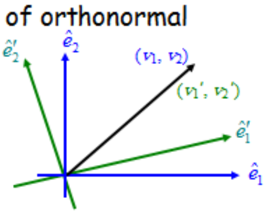
$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2$

▪Base vectors are said to be *orthonormal* if

$$\begin{cases} \hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = 1 \\ \hat{e}_1 \cdot \hat{e}_2 = 0 \end{cases}$$

▪Hence, $\hat{i}$ and $\hat{j}$ are example of orthonormal base vectors.

▪Let both $(\hat{e}_1, \hat{e}_2)$ and $(\hat{e}_1', \hat{e}_2')$ are orthonormal base vectors, i.e.,



$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 = v_1'\hat{e}_1' + v_2'\hat{e}_2'$$

$$v_1' = v_1\hat{e}_1' \cdot \hat{e}_1 + v_2\hat{e}_1' \cdot \hat{e}_2$$

$$v_2' = v_1\hat{e}_2' \cdot \hat{e}_1 + v_2\hat{e}_2' \cdot \hat{e}_2$$

How to express them in matrix form?

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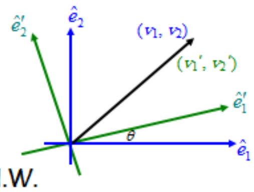
in matrix form:

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} \hat{e}_1' \cdot \hat{e}_1 & \hat{e}_1' \cdot \hat{e}_2 \\ \hat{e}_2' \cdot \hat{e}_1 & \hat{e}_2' \cdot \hat{e}_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \quad \text{....H.W.}$$



$$\begin{bmatrix} A_{x'} \\ A_{y'} \end{bmatrix} = \begin{bmatrix} \hat{i}.\hat{i}' & \hat{j}.\hat{i}' \\ \hat{i}.\hat{j}' & \hat{j}.\hat{j}' \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

Note $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are orthogonal.

Hence, an orthogonal matrix R acts as transformation to transforms a vector from one coordinates to another, i.e., $\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = R \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

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b) Properties of transformation matrix

1- Magnitude of the vectors: *Invariant* under a rotation:

$$|T\vec{v}| = |\vec{v}| = v = \sqrt{v_1^2 + v_2^2} = \sqrt{v_1'^2 + v_2'^2} = \ldots$$

2- T for a reverse rotation (-Θ) = $\tilde{T}$ (Transpose of T):

$$T(-\theta) = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \tilde{T}$$

3- $\tilde{T}T = I$, where I is the identity operator:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Ans. of Q.2:

Derivatives of Polar Unit Vectors: $\frac{d\hat{\theta}}{dt}$

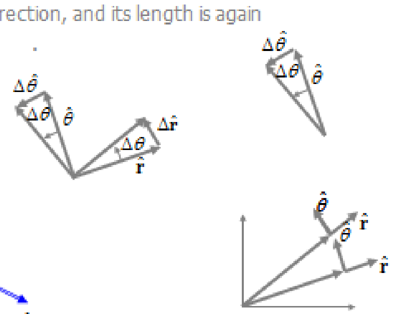
- Since the $\hat{\theta}$ unit vector is perpendicular to the $\hat{\mathbf{r}}$ unit vector, we have the same geometry as before, except rotated 90 degrees. The change $\Delta\hat{\theta}$ is now in the $-\hat{\mathbf{r}}$ direction, and its length is again $\Delta\theta = \dot{\theta}\Delta t$, so finally we have:

$$|\Delta\hat{\theta}| = \Delta\theta$$

$$\Delta\hat{\theta} = \Delta\theta(-\hat{r})$$

$$\lim_{\Delta t \to 0} \frac{\Delta\hat{\theta}}{\Delta t} = \lim_{\Delta t \to 0} [(\frac{\Delta\theta}{\Delta t}(-\hat{r})]$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$



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Ans. of Q.3:

a) *(0 seconds , <u>2 seconds</u> , 4 seconds, 6 seconds , None of them)*

b) *(-2m x^{-3} , <u>-4m x^{-3}</u> , -2m x^{-2} , -4m x^{-2} , None of them)*

c) $(\underline{\sin\theta > \mu\cos\theta}$, $\sin\theta < \mu\cos\theta$, $\sin\theta = \mu\cos\theta$, $\sin\theta \le \mu\cos\theta$, $None\ of\ them)$

d) $(Zero,\ r\ddot{\theta},\ \underline{-r\dot{\theta}^2},\ 2\dot{r}\dot{\theta}\ ,\ None\ of\ them)$

e) $[\underline{(2,\frac{\pi}{3},2)},\ (2,\frac{\pi}{6},2),\ (\sqrt{8},\frac{\pi}{3},2),\ (\sqrt{8},\frac{\pi}{6},2),\ None\ of\ them]$