Salahaddin University-Erbil College of Science Department of Mathematics 3rd Stage First Semester 2021-2022		Subject: Analytical Mechanics Period: 2 hours Date: January 2021 Final Examination First Trial

Q.1/ Express the vector $\vec{A} = 2\hat{\imath} - 4\hat{\jmath} + 6\hat{k}$ in terms of the triad $\hat{\imath}', \hat{\jmath}', \hat{k}'$ for rotation of the coordinate system about the **x-axis** through an angle $60°$. Show that the magnitude of the vector $\vec{A}$ is unchanged after the rotation. *[15 Marks]*

Q.2/ *[7+8 Marks]*

(a) For a particle moving in a circular path of radius *2m* with a velocity function $v=2t^2 m/s$. What is the magnitude of its total acceleration at $t = 1\ s$?

(b) What are the position vector, velocity and acceleration in polar coordinates?.

Q.3/ For harmonic oscillator system with: $F(x) = -kx$: *[15 Marks]*

(i) What is the solution of the system?
(ii) What are the potential energy, total energy and kinetic energy for a system?
(iii) Plot the graph of the energies of (ii) as a functions of time t and position x.

Q.4/ *[9+6 Marks]*

(A) A block is projected with initial velocity v_o on a smooth horizontal plane, but that there is air resistance proportional to v : $F(v) = -cv$, where c is constant. Find: $v(t)$, $x(t)$ and $v(x)$.

(B) Choose the **correct** answer:

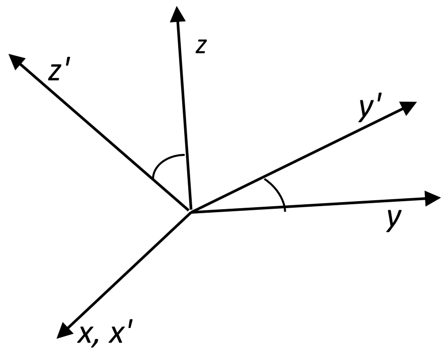
(i) What is the magnitude of the free-fall acceleration at a point that is a distance $2R_e$ above the surface of the Earth, where R_e is the radius of the Earth:
($9.8\ m/s^2$, $4.9\ m/s2$, $2.45\ m/s^2$, $1.09\ m/s^2$, *None of them*)

(ii) For an object falling from rest with linear drag. At what time the speed of the article is at 99.3% of terminal speed? ($t = 3\tau$, $t = 4\tau$, $t = 5\tau$, $t = 10\tau$, *None of them*)

(iii) What is the natural frequency for damping harmonic oscillator when the damping constant $(\beta = \omega_\circ / 2)$? ($\omega_\circ$, $\frac{\omega_\circ}{2}$, $\omega_\circ\sqrt{\frac{3}{4}}$, $\omega_\circ\sqrt{\frac{15}{16}}$, *None of them*)

Good Luck *Asst. Prof. Dr. Tahseen G. Abdullah*

Ans. of Q.1:



$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} \hat{i}.\hat{i}' & \hat{j}.\hat{i}' & \hat{k}.\hat{i}' \\ \hat{i}.\hat{j}' & \hat{j}.\hat{j}' & \hat{k}.\hat{j}' \\ \hat{i}.\hat{k}' & \hat{j}.\hat{k}' & \hat{k}.\hat{k}' \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} \cos 0 & \cos\frac{\pi}{2} & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{2} & \cos 60 & \cos(\frac{\pi}{2} - 60) \\ \cos\frac{\pi}{2} & \cos(\frac{\pi}{2} + 60) & \cos 60 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(60) & \sin(60) \\ 0 & -\sin(60) & \cos(60) \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} 2 \\ -2 + 3\sqrt{3} \\ 2\sqrt{3} + 3 \end{bmatrix} \longrightarrow \begin{aligned} A_{x'} &= 2 \\ A_{y'} &= -2 + 3\sqrt{3} = 3.196 \\ A_{z'} &= 2\sqrt{3} + 3 = 6.464 \end{aligned}$$

$$A^2 = (2)^2 + (4)^2 + (6)^2 = 4 + 16 + 36 = 56$$

$$\vec{A}' = (2)\hat{i}' + (-2 + 3\sqrt{3})\,\hat{j}' + (2\sqrt{3} + 3)\hat{k}'$$

$$\begin{aligned} A'^2 &= (2)^2 + (-2 + 3\sqrt{3})^2 + (2\sqrt{3} + 3)^2 \\ &= 4 + 4 + 27 - 12\sqrt{3} + 12 + 9 + 12\sqrt{3} \\ &= 56 = A^2 \end{aligned}$$

Ans. of Q.2:

(a) magnitude of total acceleration

$$\bar{a} = \dot{v}\hat{\tau} + \frac{v^2}{R}\hat{n} \qquad v = 2t^2 \Rightarrow \dot{v} = 4t \Rightarrow at\ t = 1s \quad v = 2m/s\ \&\ \dot{v} = 4m/s^2$$

$$a = \sqrt{\dot{v}^2 + \frac{v^4}{R^2}} = \sqrt{4^2 + \frac{2^4}{2^2}} = \sqrt{16+4} = \sqrt{20} = 4.47m/s^2$$

(b)

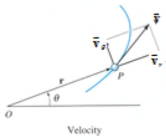
Velocity in Polar Coordinates

The instantaneous velocity is defined as:

$$\bar{\mathbf{v}} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \qquad \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

Thus, the velocity vector has two components:

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_r + \bar{\mathbf{v}}_\theta = \mathbf{v}_r\hat{r} + \mathbf{v}_\theta\hat{\theta} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$



where

$\mathbf{v}_r = \dot{r}$ …Radial component of velocity

$\mathbf{v}_\theta = r\dot{\theta}$ …Transverse component of velocity

The speed of the particle at any given instant is the sum of the squares of both components or:

$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

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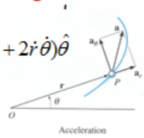
Acceleration in Polar Coordinates

The instantaneous acceleration is defined as:

$$\bar{a} = \frac{d\bar{\mathbf{v}}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{d(r\dot{\theta})}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \qquad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

After manipulation, the acceleration can be e›

$$\bar{a} = \bar{a}_r + \bar{a}_\theta = a_r\hat{r} + a_\theta\hat{\theta} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$



where $a_r = \ddot{r} - r\dot{\theta}^2$ …Radial acceleration

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ …Transverse acceleration

- r = constant (i.e. stone on a string): $\bar{a} = -r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta} = -r\omega^2\hat{r} + r\alpha\hat{\theta}$
  Here, $a_r = -r\omega^2 = -v^2/r$ is the centripetal acceleration and $a_\theta = r\alpha$ is any angular acceleration.
- When r is not constant, all terms are necessary.
- In general, the magnitude of total acceleration is:

$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$

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Ans. of Q.3:

2nd way Let the trial solution of Eq(1) is $x = e^{qt}$

Then: $\dot{x} = qe^{qt}\ \&\ \ddot{x} = q^2e^{qt}$ $\qquad \ddot{x} + \frac{k}{m}x = 0$

Substituting into Eq.(1), we obtain:

$$(q^2 + \frac{k}{m})e^{qt} = 0 \rightarrow e^{qt} \neq 0 \Rightarrow \therefore (q^2 + \frac{k}{m}) = 0 \qquad \therefore q = \pm\sqrt{-\frac{k}{m}} = \pm i\omega$$

Thus, there are two solutions of Eq(1): $e^{+i\omega t}\ \&\ e^{-i\omega t}$

The general solution is a linear combination of them:

$$x = A_1e^{i\omega t} + A_2e^{-i\omega t}$$

Or

$$x = A\cos(\omega t + \phi) \rightarrow x = x_m\cos(\omega t + \phi)$$

where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the motion

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Energy Consideration in SHM

$$x = x_m\cos(\omega t + \phi) \qquad \dot{x} = -x_m\omega\sin(\omega t + \phi)$$

1.The potential energy

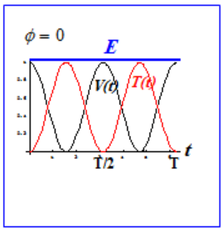
$$V = -\int F(x)dx = -\int -kxdx = \frac{1}{2}kx^2$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi)$$

2.The kinetic energy

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2x_m^2\sin^2(\omega t + \phi)$$

$$= \frac{1}{2}kx_m^2\sin^2(\omega t + \phi)$$



3. The total energy is

$$E = T + V = \frac{1}{2}kx_m^2 = \frac{1}{2}kA^2 = const.$$

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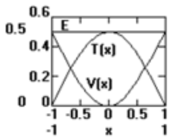
$$F(x) = -\frac{dV(x)}{dx} = -kx$$

$$V(x) = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

$$E = T_o = \frac{1}{2}kA^2$$

$$T_o = T(x) + V(x)$$

$$T(x) = T_o - V(x) = \frac{1}{2}k(A - x^2)$$



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Ans. of Q.4:

(a)

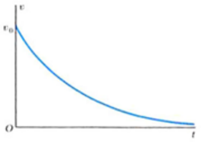
Ex.9: A block is projected with initial velocity v_o on a smooth horizontal plane, but that there is air resistance proportional to v : $F(v) = -cv$, where c is constant. Find: $v(t)$, $x(t)$ and $v(x)$. $F(v) = -cv = ma \longrightarrow a = -kv \quad (k = c/m)$

Determine the *velocity* in terms of t.

$$a = \frac{dv}{dt} = -kv \Rightarrow \frac{dv}{v} = -kdt$$

$$\Rightarrow \int_{v_0}^{v} \frac{dv}{v} = -k\int_0^t dt$$

$$\ln\left(\frac{v}{v_0}\right) = -kt \Rightarrow v = v_0 e^{-kt}$$



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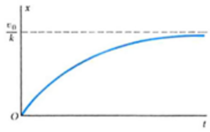
Determine the x in terms of t.

$$v = \frac{dx}{dt} = v_0 e^{-kt} \Rightarrow dx = v_0 e^{-kt} dt$$

$$\Rightarrow \int_0^x dx = \int_0^t v_0 e^{-kt} dt$$

$$x = -\frac{v_0}{k}\left[e^{-kt}\right]_0^t = \frac{v_0}{k}\left[1 - e^{-kt}\right]$$

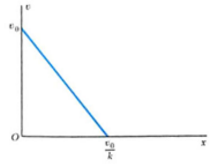


Determine the v in terms of x.

$$a = \frac{dv}{dx}\frac{dx}{dt} = -kv \Rightarrow v\frac{dv}{dx} = -kv$$

$$\Rightarrow \int_{v_0}^{v} dv = -k\int_0^x dx$$

$$v = v_0 - kx$$



$v = v(x) = dx/dt \longrightarrow x = x(t)$

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(B) Choose the **correct** answer:

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