Salahaddin University-Erbil **College of Science** **Department of Mathematics** **3rd Stage** **First Semester 2021-2022**		**Subject: Analytical Mechanics** **Period: 2 hours** **Date: June 2021** **Final Examination** **Second Trial**

Q.1/ Express the vector $\vec{A} = 4\hat{\imath} - 2\hat{\jmath}$ in terms of the triad $\hat{\imath}', \hat{\jmath}'$ for rotation of the coordinate system through an angle $60°$. What are the properties of the obtained rotation (transformation) matrix? *[15 Marks]*

Q.2/ *[5+5+5 Marks]*

(a) Define position vector and displacement of a particle.

(b) What is the corresponding point of **(1,1,1)** in cylindrical coordinates?

(c) What are the tangent and normal components of acceleration in ***n-t*** coordinates?

Q.3/ For harmonic oscillator system with: $F(x) = -kx$: *[15 Marks]*

(i) What is the solution of the system?

(ii) Plot the graph of the position, velocity and acceleration as a function of time t.

(iii) What are the potential energy, total energy and kinetic energy for a system?

(iv) Plot the graph of the energies of (iii) as a function of time t.

Q.4/ *[5+5+5 Marks]*

(A) Given the velocity of a particle in rectilinear motion varies with the displacement x according to the equation: $x = v(x) = \frac{\alpha}{x}$ where α is a positive constant. Find the force acting on the particle as a function of x?

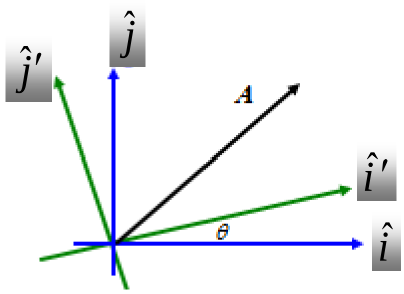
(B) What is the Mean life time in damping harmonic oscillator?

(C) For variation of gravity with height: What is the magnitude of the free-fall acceleration at a point that is a distance R_e above the surface of the Earth, where R_e is the radius of the Earth?

Good Luck *Asst. Prof. Dr. Tahseen G. Abdullah*

Ans. of Q.1:

$$\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \begin{bmatrix} \hat{i}.\hat{i}' & \hat{j}.\hat{i}' \\ \hat{i}.\hat{j}' & \hat{j}.\hat{j}' \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$



$$\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \begin{bmatrix} \cos 60 & \cos(\frac{\pi}{2} - 60) \\ \cos(\frac{\pi}{2} + 60) & \cos 60 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \begin{bmatrix} \cos 60 & \sin 60 \\ -\sin 60 & \cos 60 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2-\sqrt{3} \\ -2\sqrt{3}-1 \end{bmatrix} = \begin{bmatrix} 0.268 \\ -4.464 \end{bmatrix}$$

$$\vec{A} = A_1'\hat{i}' + A_2'\hat{j}' \qquad \vec{A} = (2-\sqrt{3})\hat{i}' + (-2\sqrt{3}-1)\hat{j}'$$

Properties of transformation matrix

1- Magnitude of the vectors: *Invariant* under a rotation:

$$|T\vec{A}| = |\vec{A}| = A = \sqrt{A_1^2 + A_2^2} = \sqrt{A_1'^2 + A_2'^2} = \ldots$$

$$A^2 = (2-\sqrt{3})^2 + (-2\sqrt{3}-1)^2$$
$$= 4+3-4\sqrt{3}+12+1+4\sqrt{3} = 20$$

$$A^2 = (4)^2 + (-2)^2$$
$$= 16+4 = 20$$

2- *T* for a reverse rotation (-*Θ*) = $\tilde{T}$ (Transpose of *T*):

$$T(-\theta) = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \tilde{T} \qquad \tilde{T} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

3- $\tilde{T}T = I$, where *I* is the identity operator:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\tilde{T}T = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

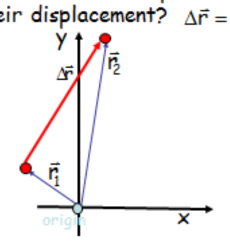
Ans. of Q.2:

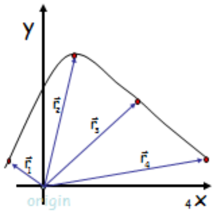
(a)

Position Vectors: A position vector is always measured from the ORIGIN.

Displacement: A change in position is called a ***displacement***. The displacement vector is typically written as $\Delta\vec{r}$

- A displacement vector is always measured from the tip of the initial position vector to the tip of the final position vector.
- A person starts at position 1 and ends at position 2, what is their displacement? $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$





(b)

Ex.: What is the corresponding point of (1,1,1) in Cylindrical Coordinates.

Sol.:

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = 45$$

$$z = z = 1$$

Thus, the corresponding point in Cylindrical Coordinates is: $(\sqrt{2}, 45, 1)$

Ex.: What is the corresponding point of $(\sqrt{2}, 45^\circ, 1)$ in Cartesian Coordinates. (H.W.)

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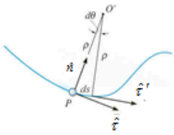
(c) What are the tangent and normal components of acceleration in ***n-t*** coordinates?

Acceleration in the (n-t) Coordinate System

Acceleration is the time rate of change of velocity:

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(v\hat{\tau}) = \dot{v}\hat{\tau} + v\frac{d\hat{\tau}}{dt}$$

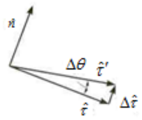


Here $\dot{v}$ represents the change in the magnitude of velocity and $\frac{d\hat{\tau}}{dt}$ represents the rate of change in the direction of $\hat{\tau}$.

$$|\Delta\hat{\tau}| = 2\sin(\Delta\theta/2)$$

$$\lim_{\Delta\theta\to 0}\frac{\Delta\hat{\tau}}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\hat{n} = \hat{n}$$

$$\frac{d\hat{\tau}}{d\theta} = \hat{n}$$



$$\frac{d\hat{\tau}}{dt} = \frac{d\hat{\tau}}{d\theta}\frac{d\theta}{dt} = \frac{d\hat{\tau}}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt} = \frac{v}{\rho}\hat{n}$$

where $\rho\, d\theta = ds$

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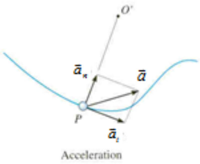

There are two components to the acceleration vector:

$$\bar{a} = \dot{v}\hat{\tau} + \frac{v^2}{\rho}\hat{n} = a_t\hat{\tau} + a_n\hat{n}$$

$$\bar{a} = \bar{a}_t + \bar{a}_n$$

- The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.
  $a_t = \dot{v}$ or $a_t\, ds = v\, dv$
- The normal or centripetal component is always directed toward the center of curvature of the curve. $a_n = v^2/\rho$
- The magnitude of the acceleration vector is
  $a = [(a_t)^2 + (a_n)^2]^{1/2}$

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Ans. of Q.3:

2nd way Let the trial solution of Eq(1) is $x = e^{qt}$

Then: $\dot{x} = qe^{qt}$ & $\ddot{x} = q^2 e^{qt}$

$$\ddot{x} + \frac{k}{m}x = 0$$

Substituting into Eq.(1), we obtain:

$$(q^2 + \frac{k}{m})e^{qt} = 0 \rightarrow e^{qt} \neq 0 \Rightarrow \therefore (q^2 + \frac{k}{m}) = 0 \qquad \therefore q = \pm\sqrt{-\frac{k}{m}} = \pm i\omega$$

Thus, there are two solutions of Eq(1): $e^{+i\omega t}$ & $e^{-i\omega t}$

The general solution is a linear combination of them:

$$x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

Or $x = A\cos(\omega t + \phi) \rightarrow \boxed{x = x_m\cos(\omega t + \phi)}$

where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the motion

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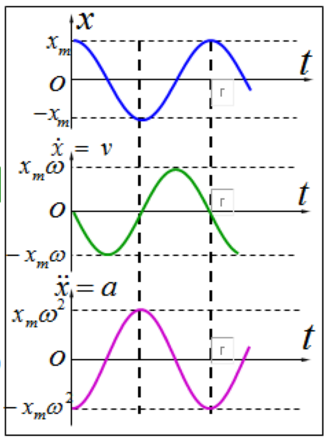
$$x = x_m\cos(\omega t + \varphi)$$

$$T = \frac{2\pi}{\omega} \qquad \varphi = 0$$

$$\dot{x} = v = -x_m\omega\sin(\omega t + \varphi)$$
$$= x_m\omega\cos(\omega t + \varphi + \frac{\pi}{2})$$

$$\ddot{x} = a = -x_m\omega^2\cos(\omega t + \varphi)$$
$$= x_m\omega^2\cos(\omega t + \varphi + \pi)$$



Spring & SHM

Energy Consideration in SHM

$x = x_m \cos(\omega t + \phi)$ | $\dot{x} = -x_m \omega \sin(\omega t + \phi)$

1.The potential energy

$$V = -\int F(x)dx = -\int -kxdx = \frac{1}{2}kx^2$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$

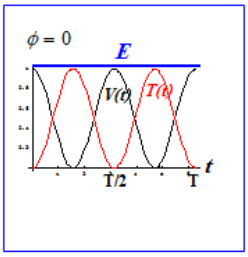
2.The kinetic energy

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$

3. The total energy is

$$E = T + V = \frac{1}{2}kx_m^2 = \frac{1}{2}kA^2 = const.$$



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Ans. of Q.4:

(A)

P.A1: Given the velocity of a particle in rectilinear motion varies with the displacement x according to the equation:

$$\dot{x} = v(x) = \frac{\alpha}{x}$$

where α is a positive constant. Find the force acting on the particle as a function of x.

$$F = m\ddot{x} = m\dot{x}\frac{d\dot{x}}{dx}$$

$$= m\frac{\alpha}{x}(-\alpha x^{-2}) = -\frac{m\alpha^2}{x^3}$$

P.4: Solve the above Problem for: $\dot{x} = v(x) = \dfrac{\alpha}{x^3}$

H.W

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(B)

Ex. What is the Mean life time in DHO?

When we add a small damping force, the amplitude gradually decreases to zero but the frequency changes by a negligible amount. In this case :

$$x(t) = Ae^{-\beta t}\cos(\omega t + \phi) = Ae^{-t/\tau}\cos(\omega t + \phi)$$

where τ is called the *"damping time constant"* or the "*mean life time*" of the oscillator:

$$\tau = \frac{1}{\beta} = \frac{2m}{c}$$

τ is the time necessary for the amplitude to drop to *1/e* of its initial value.

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(C)

Consider an object of mass m at a height h above the Earth's surface.

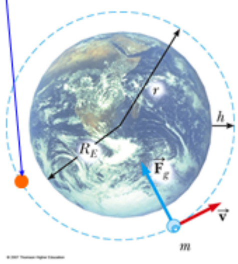
$$F = -G\frac{mM_E}{r^2} = -G\frac{mM_E}{(R_E + h)^2}$$

Acceleration g due to the gravity is:

$$F = -G\frac{mM_E}{(R_E + h)^2} = -mg$$

g will vary with altitude (height):

$$g = G\frac{M_E}{(R_E + h)^2}$$



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