Ministry of Higher Education & Scientific Research

University of Salahaddin-Erbil

College of Science

Department of Physics

2nd Year Physics

Subject: Analytical Mechanics

Chapter 5:

Dynamics of a System of Particles

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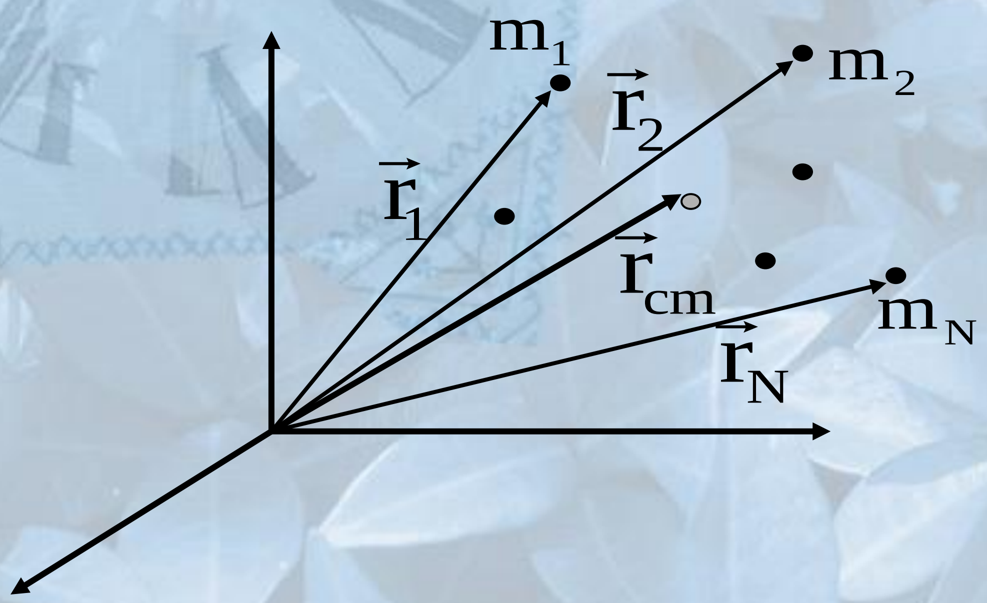
System of Particle

- **Definition:**

Consider a system consisting of N particles of masses $m_1, m_2, \ldots, m_N$. The total mass of the system is $m = \sum m_n$

Each particle can be represented by its location $\vec{r}_n$, velocity $\vec{v}_n$ and its acceleration $\vec{a}_n$.



- We can see that a system of particles behaves a lot like a particle itself
- It has a mass, position (center of mass), momentum, velocity, acceleration, and it responds to forces:

$$\vec{p} = \sum \vec{p}_n$$

- We can also define it's angular momentum, moment of the force and kinetic energy:

$$\left.\begin{aligned}\vec{L} &= \sum \left(\vec{\mathbf{r}}_n \times \vec{p}_n \right) \\ \vec{\boldsymbol{\tau}} &= \sum \left(\vec{\mathbf{r}}_n \times \vec{F}_n \right)\end{aligned}\right\} \quad \vec{\tau} = \sum \vec{\tau}_i = \frac{d\vec{L}}{dt}$$

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\vec{v}_i . \vec{v}_i \right)$$

- ## Position of the Center of Mass:

The cm of the system can be defined by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots\dots + m_N\vec{r}_N}{m_1 + m_2 + \dots\dots + m_N} = \frac{\sum_{n=1}^{N} m_n\vec{r}_n}{m} \qquad (1)$$

In terms of components, Eq(1) can be written as

$$x_{cm} = \frac{1}{m}\sum m_n x_n$$

$$y_{cm} = \frac{1}{m}\sum m_n y_n$$

$$z_{cm} = \frac{1}{m}\sum m_n z_n$$

where x_n , y_n & z_n are the coordinates of the nth particle.

Taking the derivative of the cm

$$\vec{v}_{cm} = \frac{1}{m}\sum m_n \vec{v}_n$$ ……Velocity of the cm

Differentiating once again:

$$\vec{a}_{cm} = \frac{1}{m}\sum m_n \vec{a}_n$$ ……Acceleration of the cm

Or

$$m\vec{a}_{cm} = \sum m_n \vec{a}_n = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N$$

By Newton's third law, the vector sum of all the internal forces is cancelled, and

$$\boxed{\sum \vec{F}_{ext} = m\vec{a}_{cm}}$$

This Eq. is just the Newton's second law for the system of N particles treated as a single particle of mass m located at the center of mass($\vec{r}_{cm}$), experiencing $\vec{a}_{cm}$.

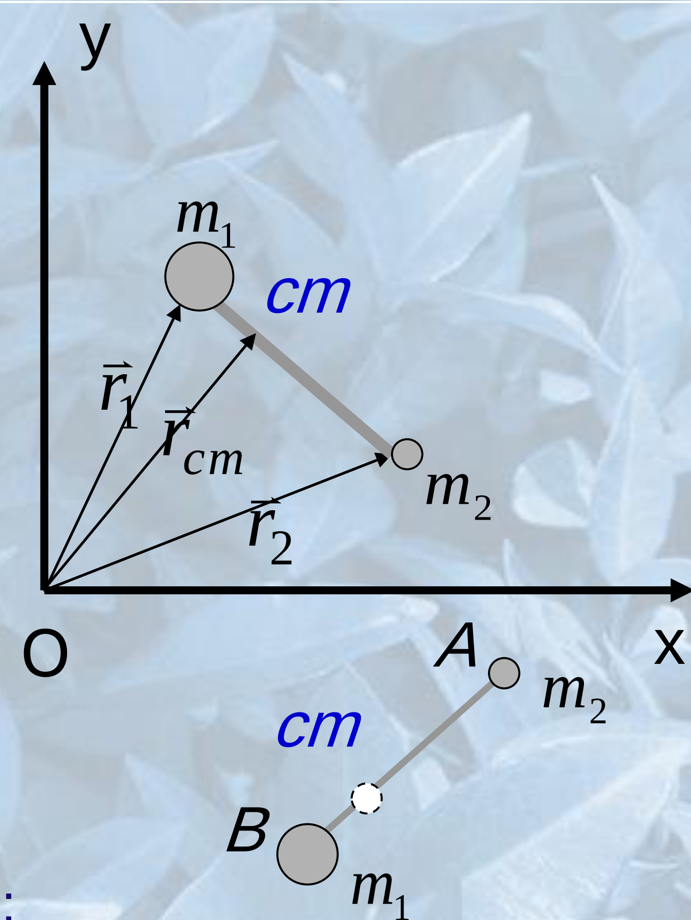
The mass center moves as if the entire mass and all of the external forces were concentrated at that point.

Example: System of two particles.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

or written as:

$$\begin{cases} x_{cm} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \\[2ex] y_{cm} = \dfrac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \end{cases}$$



The velocity and acceleration of the cm are:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \qquad \vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$$

Suppose there is an external force on each particle in above expt., then

$$m_1 \vec{a}_1 + m_2 \vec{a}_2 = \sum \vec{F}_1 + \sum \vec{F}_2 = \vec{F}_{1ext} + \vec{F}_{12} + \vec{F}_{2ext} + \vec{F}_{21}$$

$\because \vec{F}_{12} + \vec{F}_{21} = 0$, and if write $\sum \vec{F}_{1ext} + \sum \vec{F}_{2ext} = \sum \vec{F}_{ext}$

$$\Rightarrow \quad m_1\vec{a}_1 + m_2\vec{a}_2 = \sum \vec{F}_{ext} \qquad \left(\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2} \right)$$

$$\Rightarrow \quad \sum \vec{F}_{ext} = (m_1 + m_2)\vec{a}_{cm}$$

Newton's second law for system of two particles

This looks very like a particle of mass $m_1 + m_2$ located at the center of mass.

- Linear Momentum of the system:

For a system containing N particles, the total linear momentum is:

$$\vec{P} = \sum_{n=1}^{N} \vec{P}_n = \sum_{n=1}^{N} m_n \vec{v}_n = m \sum_{n=1}^{N} \frac{m_n \vec{v}_n}{m} = m\vec{v}_{cm}$$

That is, the total linear momentum of a system of particles is the total mass times the velocity of center of mass.

Differentiating w.r.t time, $\dfrac{d\vec{P}}{dt} = m\dfrac{d\vec{v}_{cm}}{dt} = m\vec{a}_{cm} = \sum \vec{F}_{ext}$

That is, the rate of change of total momentum is the net external force acting on the system.

If the net external force acting on a system is zero ($\dfrac{d\vec{P}}{dt} = 0$) and so the total linear momentum $\vec{P}$ of the system remains constant.

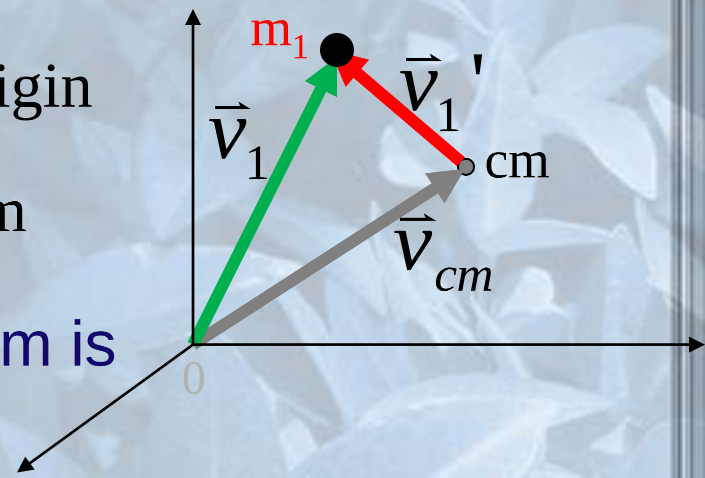
- <u>Linear Momentum of the system from the cm frame</u>

If we view the system from the cm frame, the velocity $\vec{v}_n'$ of a particle in this frame is

$$\vec{v}_n' = (\vec{v}_n - \vec{v}_{cm}) \leftarrow \begin{array}{l} \vec{v}_1' = (\vec{v}_1 - \vec{v}_{cm}) \\ \vec{v}_2' = (\vec{v}_2 - \vec{v}_{cm}) \end{array} \quad \ldots \vec{v}_N' = (\vec{v}_N - \vec{v}_{cm})$$

$\vec{v}_n$ → Velocity of particle n relative to the origin

$\vec{v}_n'$ → Velocity of particle n relative to the cm



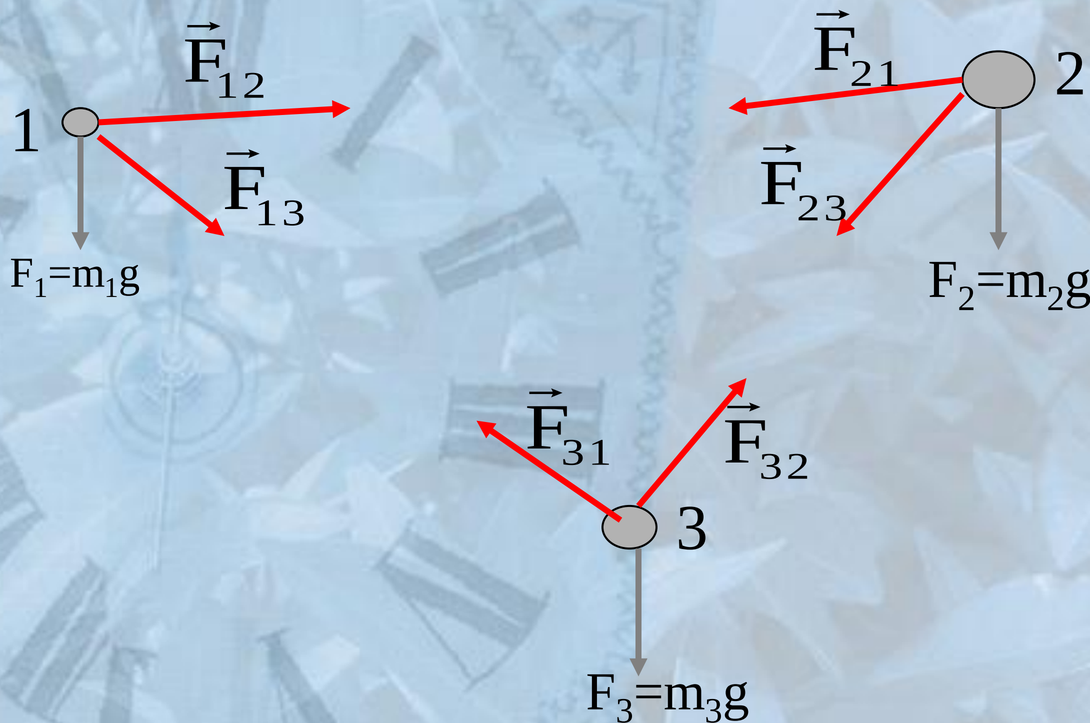
Then in this cm frame, the total momentum is

$$\vec{P}' = \sum_{n=1}^{N} m_n \vec{v}_n' = \sum_{n=1}^{N} m_n \vec{v}_n - \sum_{n=1}^{N} m_n \vec{v}_{cm}$$

$$= m\vec{v}_{cm} - m\vec{v}_{cm} = 0$$

Newton's 2^{nd} Law & Internal Forces

- If forces are generated within the particle system (say from gravity, or springs connecting particles) they must obey Newton's Third Law (every action has an equal and opposite reaction)
- This means that internal forces will balance out and have no net effect on the total momentum of the system
- As those opposite forces act along the same line of action, the torques on the center of mass cancel out as well
- In the absence of interaction among the particles, the problem is rather simple.
- One can solve the motion of each particle of the system separately.
- In the presence of interaction, the motion of the system gets enormously complicated
- With gravitational interaction, the motion of a three-body system in unsolvable.

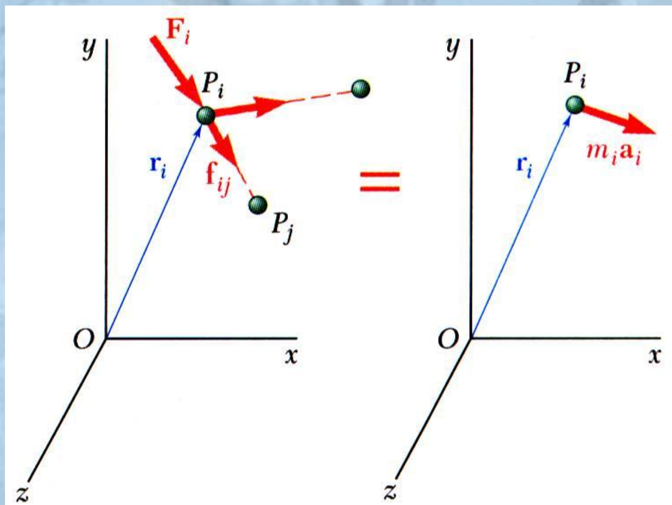


Internal forces : $\vec{F}_{12}, \vec{F}_{32}$ etc.

External forces : The weights of the particles

$$\vec{F}_1, \vec{F}_2 \ \& \ \vec{F}_3 .$$

Application of Newton's Laws. Effective Forces



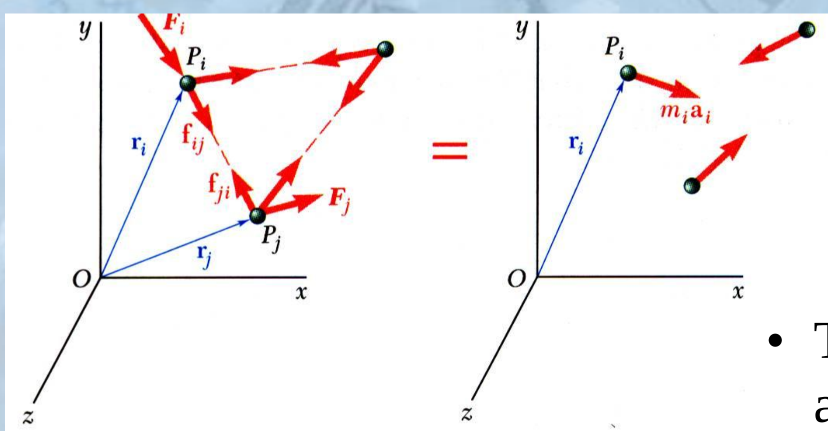
- Newton's second law for each particle P_i in a system of N particles,

$$\vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ij} = m_i \vec{a}_i$$

$$\vec{r}_i \times \vec{F}_i + \sum_{j=1}^{N} \left(\vec{r}_i \times \vec{F}_{ij} \right) = \vec{r}_i \times m_i \vec{a}_i$$

$\vec{F}_i =$ external force $\quad$ $\vec{F}_{ij} =$ internal forces

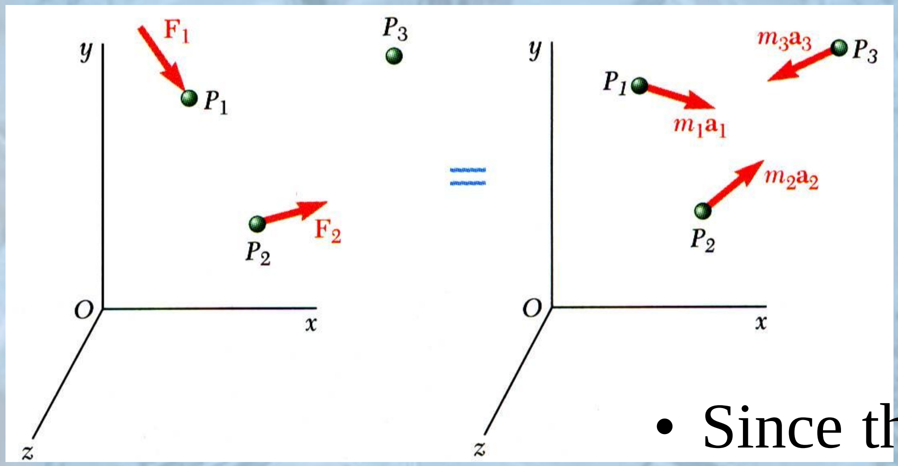
$m_i \vec{a}_i =$ effective force



The system of external and internal forces on a particle is *equivalent* to the effective force of the particle.

- The system of external and internal forces acting on the entire system of particles is *equivalent* to the system of effective forces.

- Summing over all the elements,



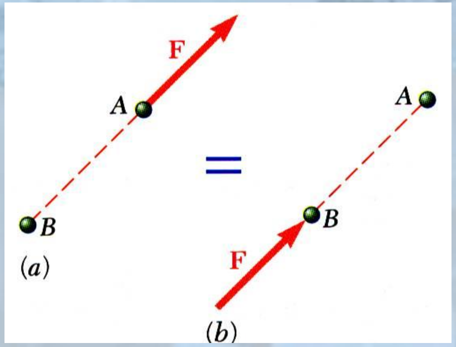
$$\sum_{i=1}^{N}\vec{F}_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ij} = \sum_{i=1}^{n} m_i\vec{a}_i$$

$$\sum_{i=1}^{N}\left(\vec{r}_i \times \vec{F}_i\right) + \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\vec{r}_i \times \vec{F}_{ij}\right) = \sum_{i=1}^{N}\left(\vec{r}_i \times m_i\vec{a}_i\right)$$

- Since the internal forces occur in equal and opposite collinear pairs, the resultant force and couple due to the internal forces are zero,

$$\sum \vec{F}_i = \sum m_i\vec{a}_i$$

$$\sum\left(\vec{r}_i \times \vec{F}_i\right) = \sum\left(\vec{r}_i \times m_i\vec{a}_i\right)$$



- The system of external forces and the system of effective forces are *equipollent* by not *equivalent.*

Angular Momentum for a System

- To calculate the total angular momentum of a system of particle about a given point, we must add vectorially the angular momenta of all the individual particles about this point:

$$\vec{L} = \sum_{i=1}^{N} \left(\vec{r}_i \times m_i \vec{v}_i \right)$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{N} \left(\dot{\vec{r}}_i \times m_i \vec{v}_i \right) + \sum_{i=1}^{N} \left(\vec{r}_i \times m_i \dot{\vec{v}}_i \right) = \sum_{i=1}^{N} \left(\vec{r}_i \times m_i \vec{a}_i \right)$$

$$= \sum_{i=1}^{N} \vec{r}_i \times \Big(\vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ij} \Big) = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i + \boxed{\sum_{i=1}^{N} \sum_{j=1}^{N} \vec{r}_i \times \vec{F}_{ij}}$$

=0 How?

- Moment resultant about fixed point O of the external forces is equal to the rate of change of angular momentum of the system of particles,

$$\vec{\tau} = \sum \vec{\tau}_i = \frac{d\vec{L}}{dt}$$

- For isolated system, the angular momentum remains constant in both magnitude and direction (Conservation of Angular Momentum).

Example: Show that the angular momentum for a system is:

$$\vec{L} = \left(\vec{r}_{cm} \times m\vec{v}_{cm}\right) + \sum_{i=1}^{N}\left(\vec{r}_i' \times m_i\vec{v}_i'\right)$$

The angular momentum of the system of particles is: $\vec{L} = \sum_{i=1}^{N}\left(\vec{r}_i \times m_i\vec{v}_i\right)$

We can express each position vector in the form:

$\vec{r}_i = \vec{r}_i' + \vec{r}_{cm}$ Differentiating w.r.t time, $\vec{v}_i = \vec{v}_i' + \vec{v}_{cm}$

$\vec{r}_i \,\&\, \vec{v}_i$ → are position & velocity of particle *i* relative to the origin

$\vec{r}_i' \,\&\, \vec{v}_i'$ → are position & velocity of particle *i* relative to the cm

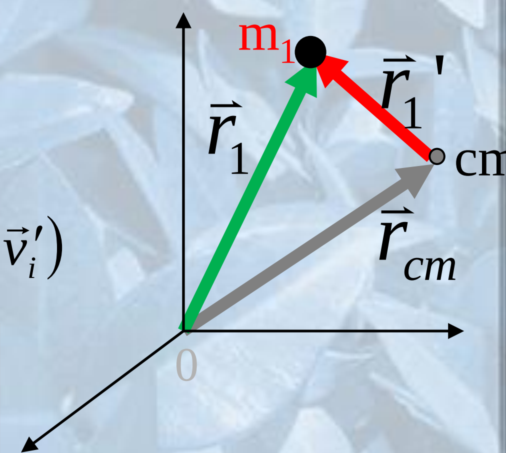
$$\vec{L} = \sum_{i=1}^{N}\left(\vec{r}_i \times m_i\vec{v}_i\right) = \sum_{i=1}^{N}\left[\left(\vec{r}_i' + \vec{r}_{cm}\right) \times m_i\left(\vec{v}_i' + \vec{v}_{cm}\right)\right]$$

$$= \sum_{i=1}^{N}\left(\vec{r}_i' \times m_i\vec{v}_i'\right) + \sum_{i=1}^{N}\left(\vec{r}_{cm} \times m_i\vec{v}_{cm}\right) + \sum_{i=1}^{N}\left(\vec{r}_i' \times m_i\vec{v}_{cm}\right) + \sum_{i=1}^{N}\left(\vec{r}_{cm} \times m_i\vec{v}_i'\right)$$

$$= \sum_{i=1}^{N}\left(\vec{r}_i' \times m_i\vec{v}_i'\right) + \left(\vec{r}_{cm} \times m\vec{v}_{cm}\right) + \boxed{\sum_{i=1}^{N}\left(m_i\vec{r}_i'\right)} \times \vec{v}_{cm} + \vec{r}_{cm} \times \boxed{\sum_{i=1}^{N}\left(m_i\vec{v}_i'\right)}$$

=0

$$\vec{L} = \left(\vec{r}_{cm} \times m\vec{v}_{cm}\right) + \sum_{i=1}^{N}\left(\vec{r}_i' \times m_i\vec{v}_i'\right) = \vec{L}_0^{cm} + \vec{L}_{cm}'$$

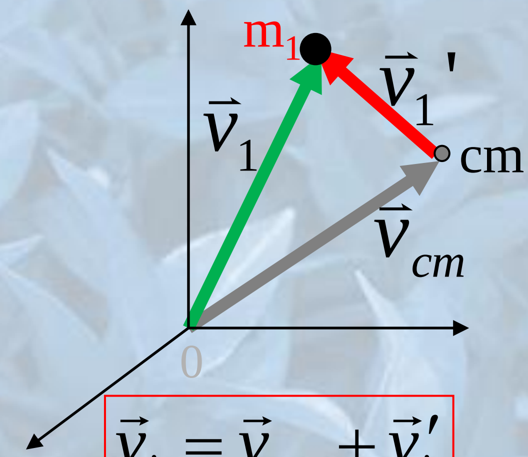


Angular momentum of the motion of cm (Orbital Part)

Angular momentum of the motion about the cm (Spin Part)

- **Kinetic Energy for a System**
  - Kinetic energy of a system of particles,

$$T=\sum_{i=1}^{N}\tfrac{1}{2}m_i v_i^2=\tfrac{1}{2}\sum_{i=1}^{N}m_i\left(\vec{v}_i\bullet\vec{v}_i\right)$$



- Expressing the velocity in terms of the cm frame,

$$\boxed{\vec{v}_i=\vec{v}_{cm}+\vec{v}_i'}$$

$$T=\tfrac{1}{2}\sum_{i=1}^{N}\left[m_i\left(\vec{v}_{cm}+\vec{v}_i'\right)\bullet\left(\vec{v}_{cm}+\vec{v}_i'\right)\right]$$

$$=\tfrac{1}{2}\left(\sum_{i=1}^{N}m_i\right)v_{cm}^2+\vec{v}_{cm}\bullet\boxed{\sum_{i=1}^{N}m_i\vec{v}_i'}^{=0}+\tfrac{1}{2}\sum_{i=1}^{N}m_i v_i'^2$$

$$=\tfrac{1}{2}mv_{cm}^2+\sum_{i=1}^{N}\tfrac{1}{2}m_i v_i'^2$$

Kinetic energy is equal to kinetic energy of mass center plus kinetic energy relative to the cm.

Kinetic energy of translation of the cm relative to the origin.

Kinetic energy of motion of the individual particles relative to the cm .

See P.7.8 & 7.11

- ## Relative motion and reduced mass

The relative motion of two particles subject only to their mutual interaction is equivalent to the motion, relative to an inertial observer, of a particle of mass equal to to the reduced mass under a force equal to their interaction.

Proof:

$$\left.\begin{aligned} \frac{d\vec{\mathbf{v}}_1}{dt} &= \frac{\vec{\mathbf{F}}_{12}}{m_1} \\ \frac{d\vec{\mathbf{v}}_2}{dt} &= \frac{\vec{\mathbf{F}}_{21}}{m_2} \end{aligned}\right\} \vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}\ , \quad \frac{d}{dt}\left(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2\right) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{\mathbf{F}}_{12}$$

with $\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_{12}$ and $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$, μ: reduced mass

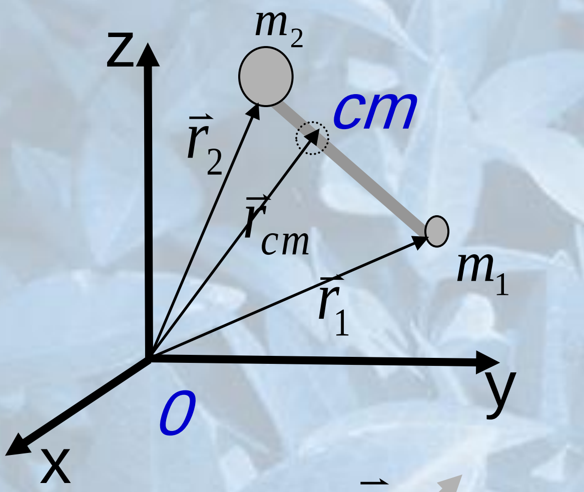
Example: sun and eath isolated or earth and moon (isolated…)

- **Motion of Two Interacting Bodies. Two Body Problem**

Let us consider the motion of a system consisting of two bodies that interact with one another by a central force.
For isolated system:

$\vec{v}_{cm} = const.$ i.e. center of mass moves with constant velocity

$$\vec{r}_{cm} = \frac{1}{m}\sum_{i=1}^{2} m_i \vec{r}_i = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$



For simplicity we take the cm at the origin:

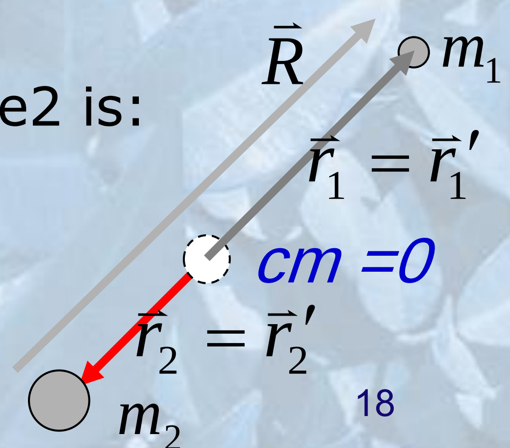
$$\vec{r}_{cm} = 0 \Rightarrow m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

$$m_1\vec{r}_1{}' + m_2\vec{r}_2{}' = 0 \Rightarrow \vec{r}_2{}' = -\frac{m_1}{m_2}\vec{r}_1{}'$$

The position vector of particle1 relative to particle2 is:

$$\boxed{\vec{R} = \vec{r}_1{}' - \vec{r}_2{}'} = \vec{r}_1{}'(1 + \frac{m_1}{m_2}) = \vec{r}_1{}'(\frac{m_1 + m_2}{m_2})$$

$$\vec{r}_1{}' = (\frac{m_2}{m_1 + m_2})\vec{R} \Rightarrow \ddot{\vec{r}}_1{}' = (\frac{m_2}{m_1 + m_2})\ddot{\vec{R}}$$



The differential equation of motion of particle1 relative to the center of mass is given by Newton's 2nd law:

$$m_1\ddot{\vec{r}}_1' = \vec{F}_1 = f(R)\frac{\vec{R}}{R} \quad \Rightarrow \quad \frac{m_1 m_2}{m_1+m_2}\ddot{\vec{R}} = f(R)\frac{\vec{R}}{R}$$

or

$$\boxed{\mu\ddot{\vec{R}} = f(R)\frac{\vec{R}}{R}}$$

...motion of particle1 relative to particle2. (Motion of Central Field)

where

$$\mu = \frac{m_1 m_2}{m_1+m_2} \Rightarrow$$

...is the reduced mass

$f(R) \Rightarrow$ is the magnitude of the mutual force between the two bodies. For two bodies attracting one another by gravitation, we have:

$$f(R) = -G\frac{m_1 m_2}{R^2} \quad \Rightarrow \quad \boxed{\mu\ddot{\vec{R}} = -G\frac{m_1 m_2}{R^2}\left(\frac{\vec{R}}{R}\right)}$$

This is the same as the equation of a single particle in an (Inverse-Square-Central field)

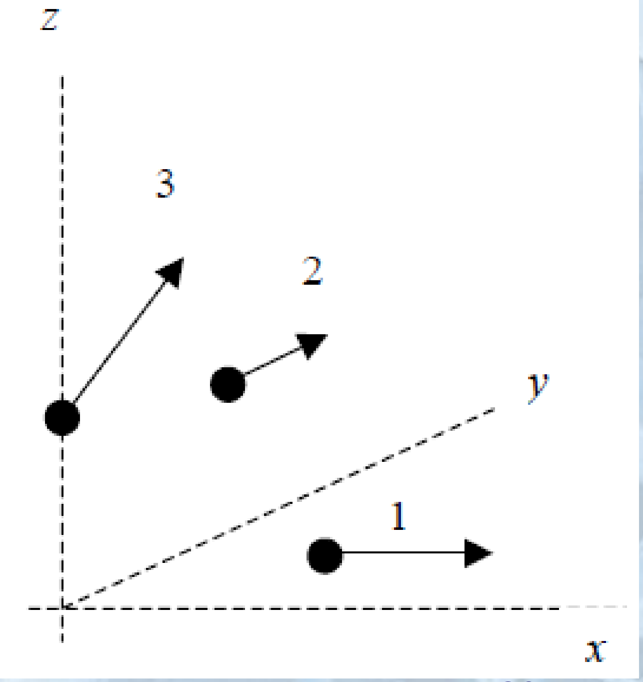
P.7.1: A system consists of three particles, each of unit mass, with positions and velocities as follows:

$$\vec{r}_1 = \hat{i} + \hat{j} \qquad \vec{v}_1 = 2\hat{i}$$

$$\vec{r}_2 = \hat{j} + \hat{k} \qquad \vec{v}_2 = \hat{j}$$

$$\vec{r}_3 = \hat{k} \qquad \vec{v}_3 = \hat{i} + \hat{j} + \hat{k}$$

Find the position and velocity of the center of mass. Find also the linear momentum of the system.



Solution

$$\vec{r}_{cm} = \frac{1}{m}\sum_i m_i \vec{r}_i$$

$$\vec{r}_{cm} = \frac{1}{3}\left(\vec{r}_1 + \vec{r}_2 + \vec{r}_3\right) = \frac{1}{3}\left(\hat{i} + \hat{j} + \hat{j} + \hat{k} + \hat{k}\right)$$

$$\vec{r}_{cm} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

$$\vec{v}_{cm} = \frac{d}{dt}\vec{r}_{cm} = \frac{1}{3}\left(\vec{v}_1 + \vec{v}_2 + \vec{v}_3\right) = \frac{1}{3}\left(2\hat{i} + \hat{j} + \hat{i} + \hat{j} + \hat{k}\right)$$

$$\vec{v}_{cm} = \frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$$

$$\vec{P} = \sum_{n=1}^{N} \vec{P}_n = \sum_{n=1}^{N} m_n \vec{v}_n = m \sum_{n=1}^{N} \frac{m_n \vec{v}_n}{m} = m\vec{v}_{cm}$$

$$\vec{p} = \sum_i m_i \vec{v}_i = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$$

$$\vec{p} = 3\hat{i} + 2\hat{j} + \hat{k}$$

P.7.2: Find also from P.7.1: (a) Kinetic energy for a system (b) Kinetic energy of the center of mass relative to the origin and (c) Angular momentum of the system.

(a) $$T = \sum_i \frac{1}{2} m_i v_i^2$$

$$T = \frac{1}{2}\left[2^2 + 1^2 + \left(1^2 + 1^2 + 1^2\right)\right] = 4$$

(b) $$\vec{v}_{cm} = \frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$$

$$\frac{1}{2} m v_{cm}^2 = \frac{1}{2} \times 3 \times \frac{1}{9}\left(3^2 + 2^2 + 1^2\right) = 2\frac{1}{3}$$

(c) $$\vec{L} = \sum_{i=1}^{N}\left(\vec{r}_i \times m_i \vec{v}_i\right)$$

$$\vec{L} = \left[\left(\hat{i} + \hat{j}\right) \times 2\hat{i}\right] + \left[\left(\hat{j} + \hat{k}\right) \times \hat{j}\right] + \left[\hat{k} \times \left(\hat{i} + \hat{j} + \hat{k}\right)\right]$$

$$\vec{L} = \left(-2\hat{k}\right) + \left(-\hat{i}\right) + \left(\hat{j} - \hat{i}\right) = -2\hat{i} + \hat{j} - 2\hat{k}$$

P.7.8: Show that the kinetic energy of a two particle system is equal to:

$$T = \tfrac{1}{2} m v_{cm}^2 + \tfrac{1}{2} \mu v^2$$

Where $m = m_1 + m_2$ $,v$ is the related speed before collision and is μ the reduced mass . (H.W.)

P.7.11: Show that the angular momentum of a two particle system is equal to:

$$\vec{L} = \left(\vec{r}_{cm} \times m\vec{v}_{cm}\right) + \vec{R} \times \mu\vec{v}$$

Where $m = m_{1'} + m_2$

$\vec{R}$ is the relative position vector, μ is the reduced mass

and $\vec{v}$ is the relative velocity of the two particles. (H.W.)

- <u>Solution of P. 7.8:</u>

$$T=\sum_i \frac{1}{2}m_i v_i^2=\frac{1}{2}m_1 v_1^2+\frac{1}{2}m_2 v_2^2$$

Meanwhile:

$$\frac{1}{2}mv_{cm}^2+\frac{1}{2}\mu v^2=\frac{1}{2}m\left(\frac{m_1\vec{v}_1+m_2\vec{v}_2}{m}\right)^2+\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(\vec{v}_1-\vec{v}_2\right)^2$$

$$=\frac{1}{2m}\left[m_1^2 v_1^2+m_2^2 v_2^2+2m_1 m_2\vec{v}_1\cdot\vec{v}_2+m_1 m_2\left(v_1^2+v_2^2-2\vec{v}_1\cdot\vec{v}_2\right)\right]$$

$$=\frac{1}{2}m_1 v_1^2+\frac{1}{2}m_2 v_2^2$$

Therefore, $T=\frac{1}{2}mv_{cm}^2+\frac{1}{2}\mu v^2$

- <u>Solution of P. 7.11:</u>

$$\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \sum_i \vec{\bar{r}}_i \times m_i \vec{\bar{v}}_i$$

$$\sum_i \vec{\bar{r}}_i \times m_i \vec{\bar{v}}_i = \vec{\bar{r}}_1 \times m_1 \vec{\bar{v}}_1 + \vec{\bar{r}}_2 \times m_2 \vec{\bar{v}}_2$$

From eqn. 7.3.2, $\vec{R} = \vec{\bar{r}}_1\left(1 + \frac{m_1}{m_2}\right) = \vec{\bar{r}}_1\left(\frac{m_1 + m_2}{m_2}\right) = \frac{m_1}{\mu}\vec{\bar{r}}_1$

Since from eqn. 7.3.1, $\vec{\bar{r}}_1 = -\frac{m_2}{m_1}\vec{\bar{r}}_2$

$$\vec{R} = -\frac{m_2}{\mu}\vec{\bar{r}}_2$$

$$\sum_i \vec{\bar{r}}_i \times m_i \vec{\bar{v}}_i = \frac{\mu}{m_1}\vec{R} \times m_1 \vec{\bar{v}}_1 + \left(-\frac{\mu}{m_2}\right)\vec{R} \times m_2 \vec{\bar{v}}_2$$

$$= \mu\vec{R} \times \left(\vec{\bar{v}}_1 - \vec{\bar{v}}_2\right) = \vec{R} \times \mu\vec{v}$$

$$\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \vec{R} \times \mu\vec{v}$$

Collisions

- Collision in 1-D (Direct or Head-on Collision)
- Collision in 2-D (Oblique Collision)

Basic Facts

Students will:

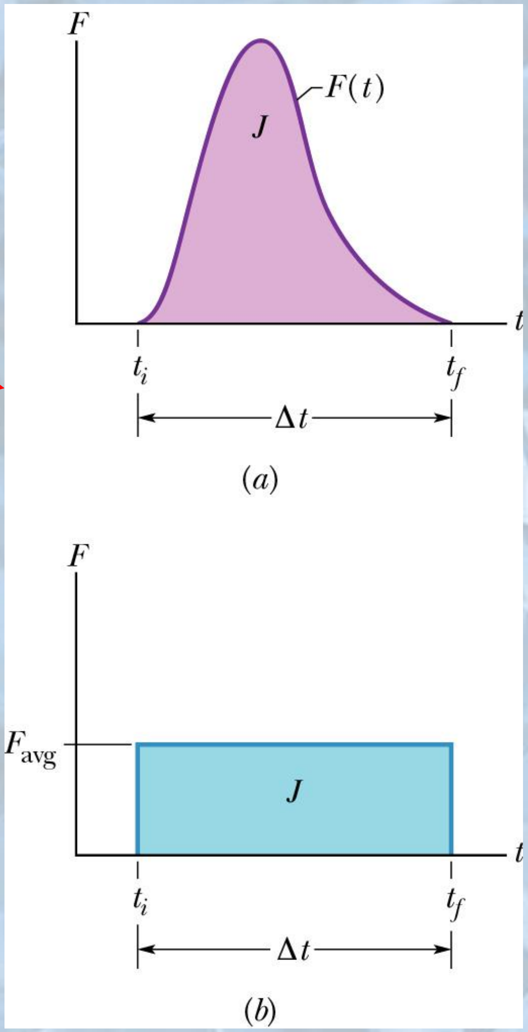
- Identify different types of collisions;
- Determine the changes in kinetic energy during perfectly inelastic collisions
- Compare conservation of momentum and conservation of kinetic energy in perfectly inelastic and elastic collisions.
- Find the final velocity of an object in perfectly inelastic and elastic collisions.

Definition of Impulse

$$d\vec{p} = \vec{F}(t)dt$$

$$\int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt$$



Impulse-Momentum Theorem :

The theorem states that the impulse acting on a system is equal to the change in momentum of the system

$$\Delta\vec{p} = \vec{F}_{net}\Delta t = \vec{J}$$

$$\boxed{\vec{J} = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i}$$

Calculating the Change of Momentum

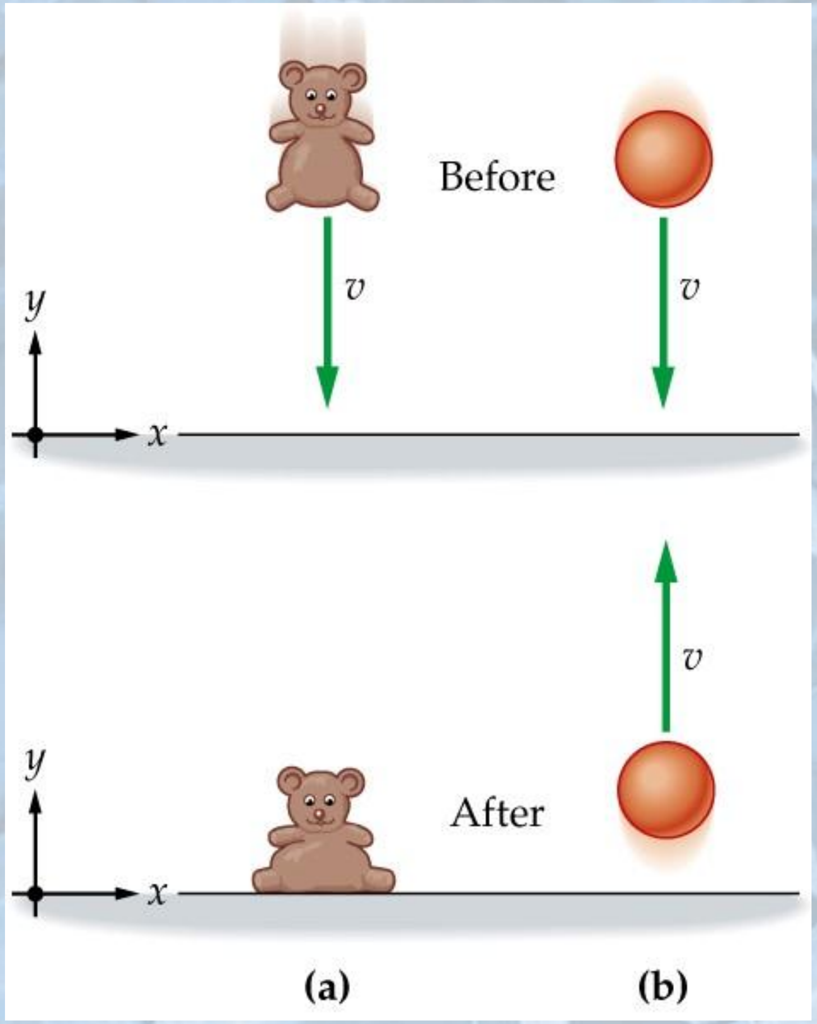
$$\Delta\vec{p} = \vec{p}_{after} - \vec{p}_{before}$$
$$= mv_{after} - mv_{before}$$
$$= m(v_{after} - v_{before})$$

For the teddy bear

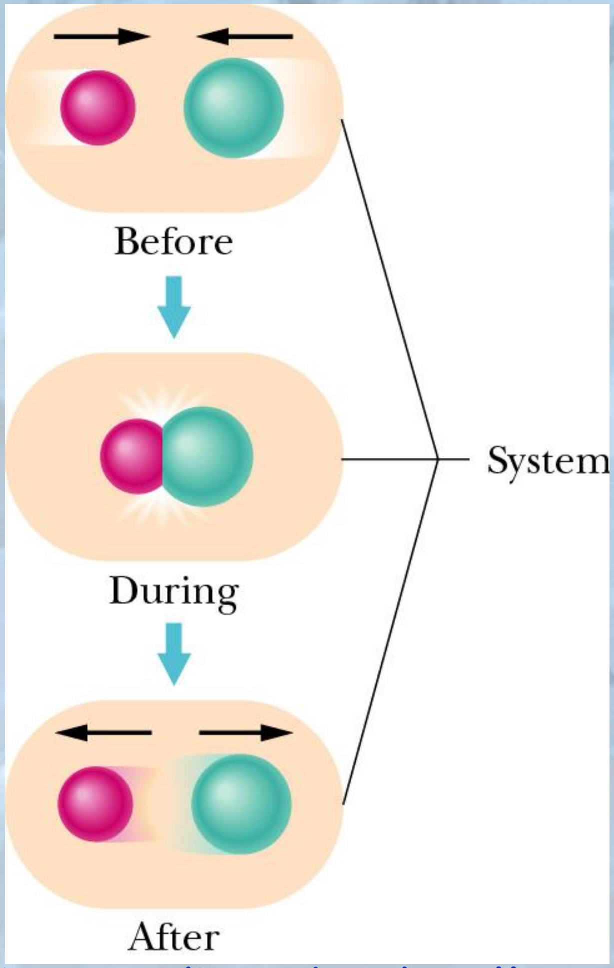
$$\Delta p = m[0-(-v)] = mv$$

For the bouncing ball

$$\Delta p = m[v-(-v)] = 2mv$$



Conservation of Momentum

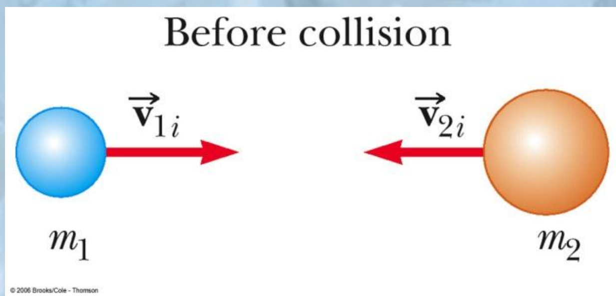


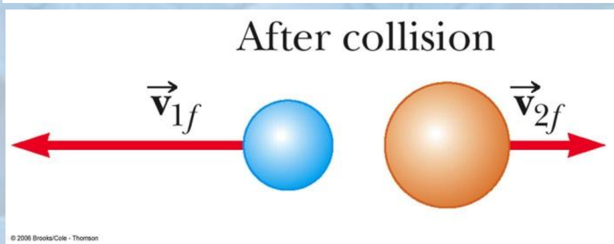
Direct (Head-on) collision

- In an isolated and closed system, the total momentum of the system remains constant in time.
  - Isolated system: no external forces
  - Closed system: no mass enters or leaves
  - The linear momentum of each colliding body may change
  - The total momentum of the system cannot change.

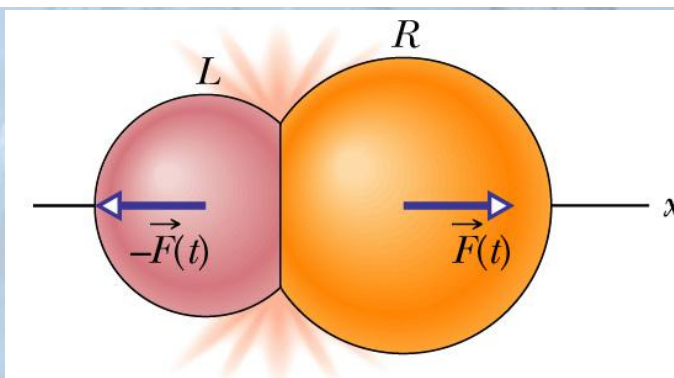
$$\vec{p}_{before} = \vec{p}_{after} \rightarrow \vec{p} = \vec{p}' \rightarrow \vec{p}_i = \vec{p}_f$$

Ex. Satisfy Conservation of Momentum from impulse-momentum theorem





Direct (Head-on) collision



- Start from impulse-momentum theorem

$$\vec{F}_{21}\Delta t = m_1\vec{v}_{1f} - m_1\vec{v}_{1i}$$

$$\vec{F}_{12}\Delta t = m_2\vec{v}_{2f} - m_2\vec{v}_{2i}$$

- Since $\vec{F}_{21}\Delta t = -\vec{F}_{12}\Delta t$
- Then $m_1\vec{v}_{1f} - m_1\vec{v}_{1i} = -(m_2\vec{v}_{2f} - m_2\vec{v}_{2i})$
- So $$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\vec{p}_{before} = \vec{p}_{after} \rightarrow \vec{p}_i = \vec{p}_f$$

Types of Collisions

- Momentum is conserved in any collision
- **Inelastic collisions:** *rubber ball and hard ball*
  - Kinetic energy is not conserved
  - **Perfectly inelastic collisions** occur when the objects stick together
- **Elastic collisions:** *billiard ball*
  - both momentum and kinetic energy are conserved
- **Actual collisions**
  - Elastic and perfectly inelastic collisions are limiting cases
  - Most collisions fall between elastic and perfectly inelastic collisions

Elastic Collisions

An Elastic collision is a collision in which the total momentum and the total kinetic energy remain constant.

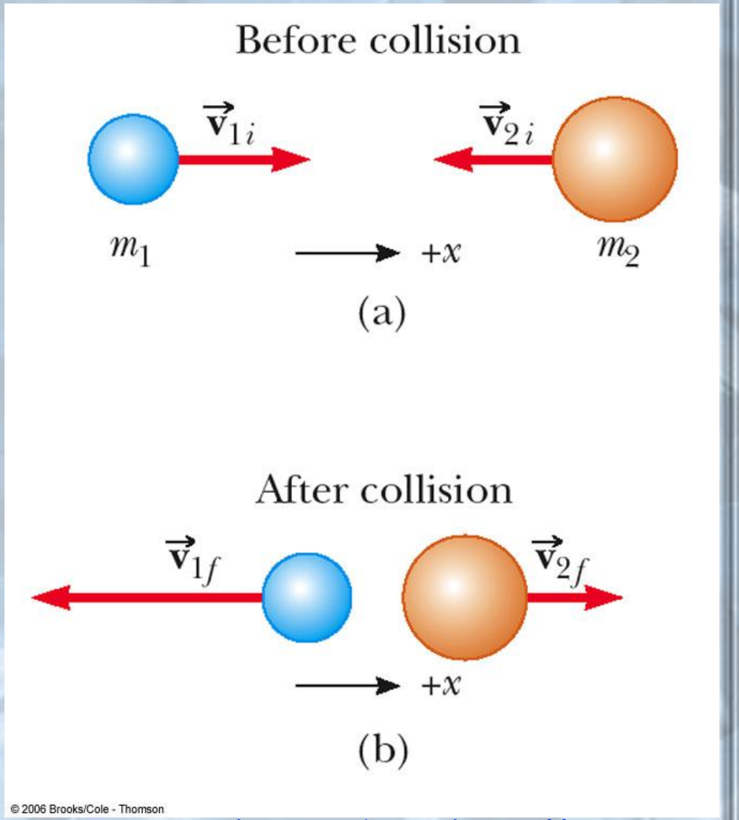
Momentum

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Kinetic Energy

$$\tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

$$\frac{p_{1i}^2}{2m_1} + \frac{p_{2i}^2}{2m_2} = \frac{p_{1f}^2}{2m_1} + \frac{p_{2f}^2}{2m_2}$$



Direct (Head-on) collision

- A simpler equation can be used in place of the kinetic energy equation:

$$\tfrac{1}{2}m_1v_{1i}^2+\tfrac{1}{2}m_2v_{2i}^2=\tfrac{1}{2}m_1v_{1f}^2+\tfrac{1}{2}m_2v_{2f}^2$$

$$m_1(v_{1i}^2-v_{1f}^2)=m_2(v_{2f}^2-v_{2i}^2)$$

$$m_1(v_{1i}-v_{1f})(v_{1i}+v_{1f})=m_2(v_{2f}-v_{2i})(v_{2f}+v_{2i})$$

$$\boxed{m_1v_{1i}+m_2v_{2i}=m_1v_{1f}+m_2v_{2f}} \rightarrow m_1(v_{1i}-v_{1f})=m_2(v_{2f}-v_{2i})$$

$$\boxed{(v_{1i}+v_{1f})=(v_{2f}+v_{2i})} \longrightarrow \boxed{v_{1f}-v_{2f}=v_{2i}-v_{1i}}$$

Speed of separation

Speed of approach

$$\boxed{\frac{v_{1f}-v_{2f}}{v_{2i}-v_{1i}}}=1$$

Coefficient of restitution for elastic collision

$$\boxed{\varepsilon=1}$$

Inelastic Collisions

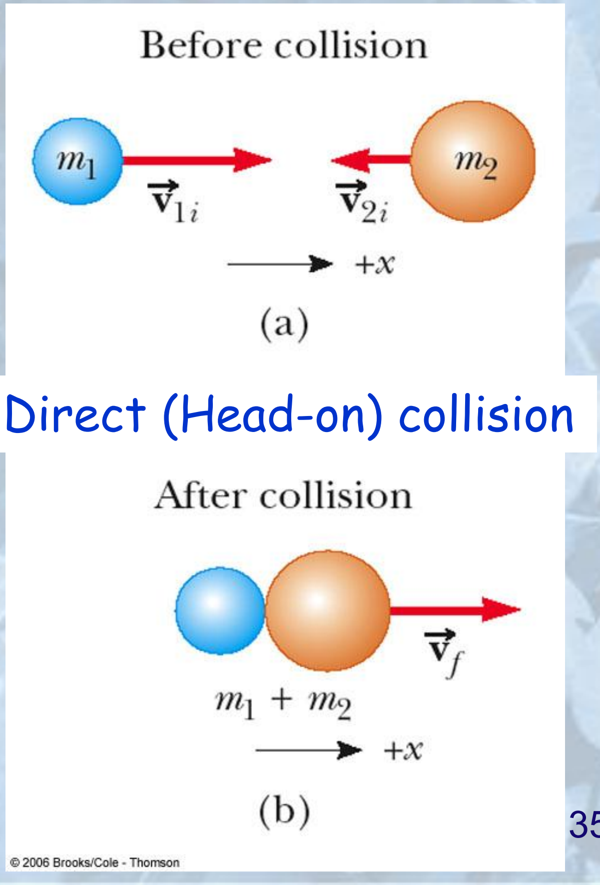
Inelastic collision is a collision in which momentum is conserved but kinetic energy is not. Moreover, the objects do not stick together.

Kinetic Energy is not constant in inelastic collisions. Some kinetic energy is converted to sound and/or heat, or causes deformation. To calculate the amount of kinetic energy that is lost,

$$T_i = \tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2$$

$$T_f = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

$$\Delta T = T_f - T_i$$



Perfectly inelastic collision is a collision in which two objects stick together and move with a common velocity after colliding.

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

Summary of Collisions

- In general, for a system of two bodies undergo a collision, the total linear momentum and total kinetic energy are:

$$\boxed{\vec{p}_{befor} = \vec{p}_{after} \quad \Rightarrow \quad m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}}$$

$$\boxed{\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 + Q}$$

Q is the net energy loss or gain in kinetic energy that occurs as a result of collision

$Q=0$ …..Elastic collision (there is no change in kinetic energy)

$Q=+ive$ …..Endoergic collision (there is an energy loss)

$Q=-ive$ …..Exoergic collision (there is an energy gain)

Velocity of particles after collision

- Now, we can find the velocities of particle1 and particle2 relative to their velocities before collision. For a head-on collision we have:

$$\boxed{m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}}$$ …Conservation of Momentum

$$\varepsilon = \frac{v_{1f} - v_{2f}}{v_{2i} - v_{1i}}$$ …Coefficient of restitution $\rightarrow$ $$\boxed{v_{1f} - v_{2f} = \varepsilon\left(v_{2i} - v_{1i}\right)}$$

$$\boxed{v_{1f} = \frac{(m_1 - m_2\varepsilon)v_{1i} + (m_2 + m_2\varepsilon)v_{2i}}{m_1 + m_2}}$$

$$\boxed{v_{2f} = \frac{(m_1 + m_1\varepsilon)v_{1i} + (m_2 - m_1\varepsilon)v_{2i}}{m_1 + m_2}}$$

From the above equations, we have the following cases:

- In an elastic collision, $\varepsilon=1$ and in the special case when $m_1=m_2$ we obtain: $$v_{1f}=v_{2i} \;\; \& \;\; v_{2f}=v_{1i}$$

  Therefore, the two bodies just exchange their velocities as a result of collision

- In a perfectly inelastic collision, $\varepsilon=0$ and $v_{1f}=v_{2f}$

  Thus the two objects stick together after the collision, so their final velocities are the same

- In the general case of a direct inelastic collision, $\varepsilon=(0\text{-}1)$ and the energy loss Q is related to ε by:

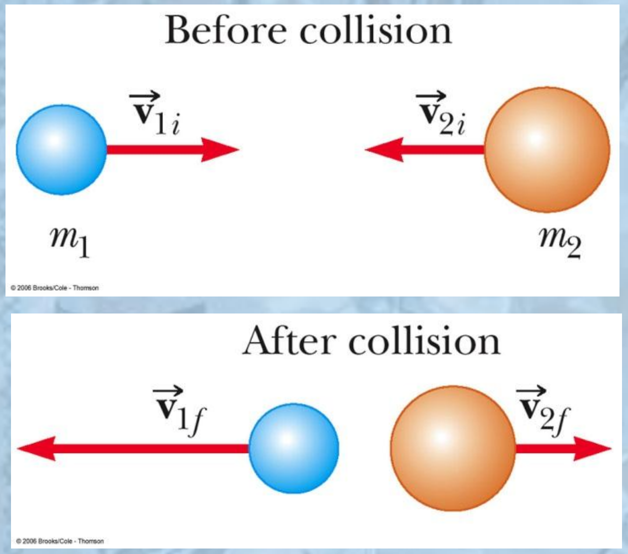
  $$Q=\frac{1}{2}\mu v^2(1-\varepsilon^2)$$

  where

  μ is the reduced mass

  v is the related speed before collision

Comparison One-Dimensional & Two-Dimensional Collisions



Direct (Head-on) Collision

Oblique Collision

Two-Dimensional (Oblique) Collisions

- For a general collision of two objects in two-dimensional space, the conservation of momentum principle implies that the *total momentum of the system in each direction is conserved*

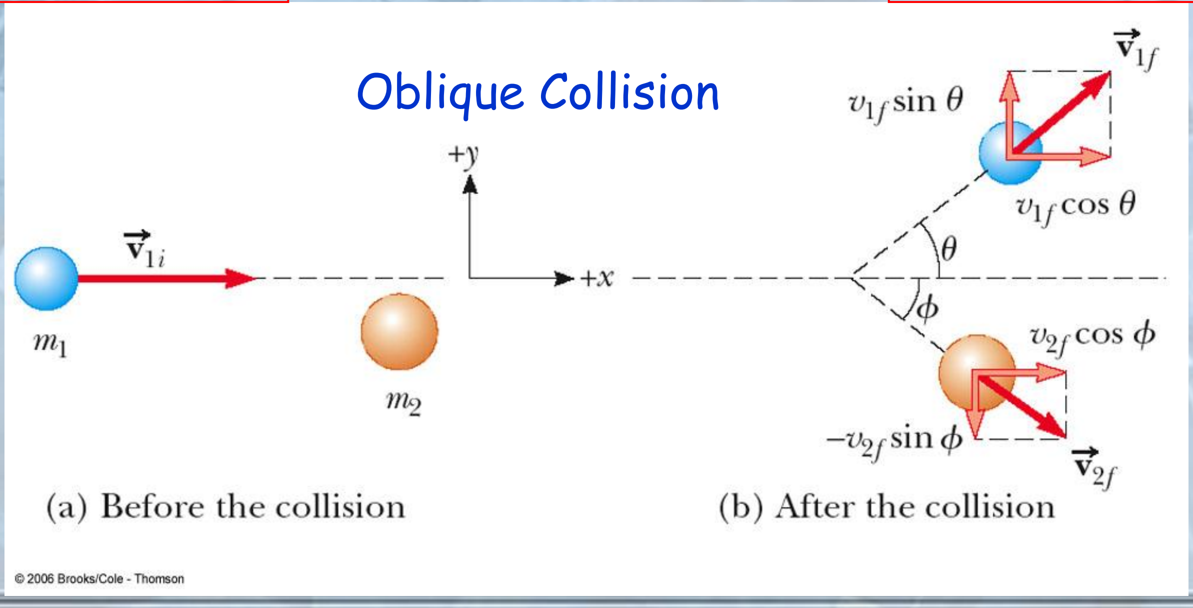
$$\vec{p}_{befor} = \vec{p}_{after} \quad \Rightarrow \quad m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

x-direction

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

y-direction



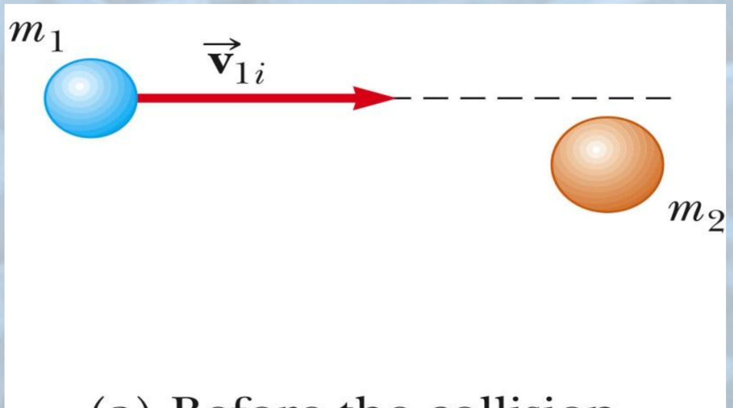
In Oblique Collisions:

- The momentum is conserved in the x direction and in the y direction. Apply conservation of momentum separately to each direction.
- If the collision is elastic, use conservation of kinetic energy as a second equation
  - Remember, the simpler equation can only be used for one-dimensional situations.

$$\cancel{\left(v_{1i} + v_{1f}\right) = \left(v_{2f} + v_{2i}\right)}$$

Ex.: Oblique Collision (2-D Collision)

- Particle 1 is moving at velocity $\vec{\mathbf{v}}_{1i}$ and particle 2 is at rest.
- In the *x*-direction, the initial momentum is $m_1 v_{1i}$
- In the *y*-direction, the initial momentum is 0



(a) Before the collision

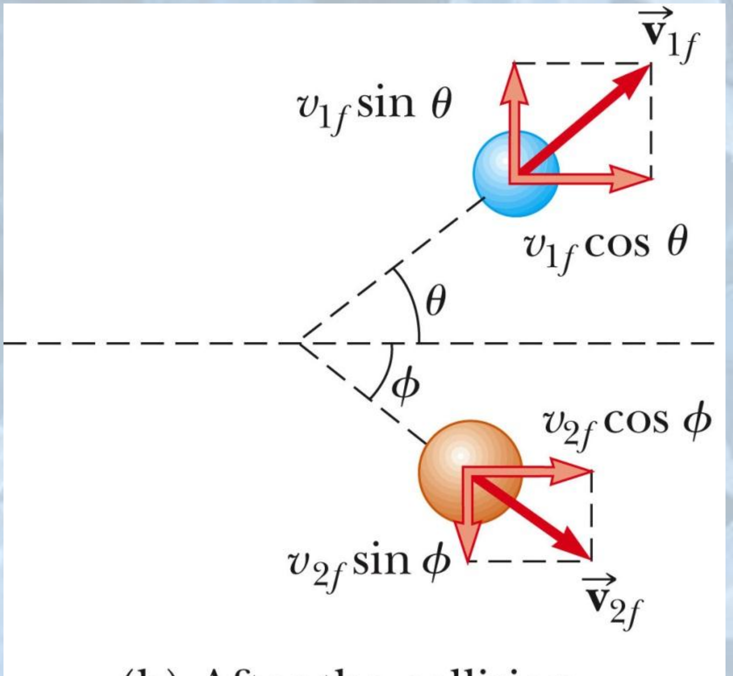
- After the collision, the momentum in the *x*-direction is: $m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi$
- After the collision, the momentum in the *y*-direction is: $m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi$

$$m_1 v_{1i} + 0 = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\varphi$$

$$0 + 0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\varphi$$

- If the collision is elastic, apply the kinetic energy equation

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$



(b) After the collision

- If the collision is inelastic, the kinetic energy equation is

$$\frac{1}{2}m_1v_{1i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 + Q \rightarrow \frac{p_{1i}^2}{2m_1} = \frac{p_{1f}^2}{2m_1} + \frac{p_{2f}^2}{2m_2} + Q$$

- The momentum equation in vector notation can be written as:

$$m_1\vec{v}_{1i} + 0 = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} \rightarrow \vec{p}_{1i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$p_{1i}^2 = (\vec{p}_{1f} + \vec{p}_{2f}) \bullet (\vec{p}_{1f} + \vec{p}_{2f}) = p_{1f}^2 + p_{2f}^2 + 2(\vec{p}_{1f} \bullet \vec{p}_{2f})$$

- In the case, when the masses of the incident and target particles are the same the kinetic energy equation becomes:

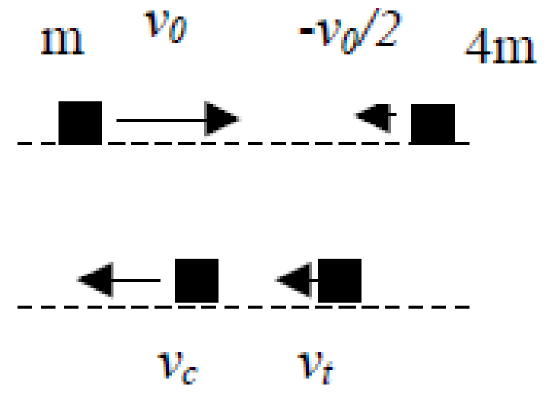
$$p_{1i}^2 = p_{1f}^2 + p_{2f}^2 + 2mQ$$

$$\boxed{Q = \frac{(\vec{p}_{1f} \bullet \vec{p}_{2f})}{m}}$$

For elastic collision *Q=0*:

$$\boxed{\vec{p}_{1f} \bullet \vec{p}_{2f} = 0 \Rightarrow \vec{p}_{1f} \perp \vec{p}_{2f}}$$

P.7.7: A small car of mass ***m*** and initial speed v_0 collides head-on on an icy road with a truck of mass ***4m*** going toward the car with initial speed $v_0/2$. If the coefficient of restitution in the collision is ¼ , find the speed and direction of each vehicle just after colliding.



$$v_{1f} = \frac{(m_1 - m_2\varepsilon)v_{1i} + (m_2 + m_2\varepsilon)v_{2i}}{m_1 + m_2}$$

$$v_{2f} = \frac{(m_1 + m_1\varepsilon)v_{1i} + (m_2 - m_1\varepsilon)v_{2i}}{m_1 + m_2}$$

$$v_c = \frac{\left(m - \frac{1}{4}4m\right)v_\circ + \left(4m + \frac{1}{4}4m\right)\left(-\frac{v_\circ}{2}\right)}{m + 4m} = \frac{0 + 5m\left(-\frac{v_\circ}{2}\right)}{5m} = -\frac{v_\circ}{2}$$

$$v_t = \frac{\left(m + \frac{1}{4}m\right)v_\circ + \left(4m - \frac{1}{4}m\right)\left(-\frac{v_\circ}{2}\right)}{m + 4m}$$

$$= \frac{\frac{5}{4}mv_\circ + \frac{15}{4}m\left(-\frac{v_\circ}{2}\right)}{5m} = -\frac{v_\circ}{8}$$

Both car & truck are traveling in the initial direction of the truck with speeds $v_0/2$ & $v_0/8$, respectively.

P.7.9: If two bodies undergo a direct collision, show that the loss in kinetic energy is equal to:

$$Q = \frac{1}{2}\mu v^2 (1-\varepsilon^2)$$

where

μ is the reduced mass

v is the related speed before collision

ε Is the coefficient of restitution

$$T = \tfrac{1}{2} m v_{cm}^2 + \tfrac{1}{2}\mu v^2$$

... Kinetic Energy of a two particle system before collision.

$$T' = \tfrac{1}{2} m v_{cm}^2 + \tfrac{1}{2}\mu v'^2$$

... Kinetic Energy of a two particle system after collision.

$Q = T - T'$ and since $v_{cm} = v'_{cm}$:

$$Q = \frac{1}{2}\mu v^2 - \frac{1}{2}\mu v'^2$$

$$\varepsilon = \frac{v'}{v}$$

$$Q = \frac{1}{2}\mu v^2 \left(1 - \varepsilon^2\right)$$

End of the Lecture

Let Learning Continue

Thank You