Ministry of Higher Education & Scientific Research

University of Salahaddin-Erbil

College of Science

Department of Mathematics

3rd Stage

Subject: Analytical Mechanics

Chapter 7:

Kepler's laws of planetary motion

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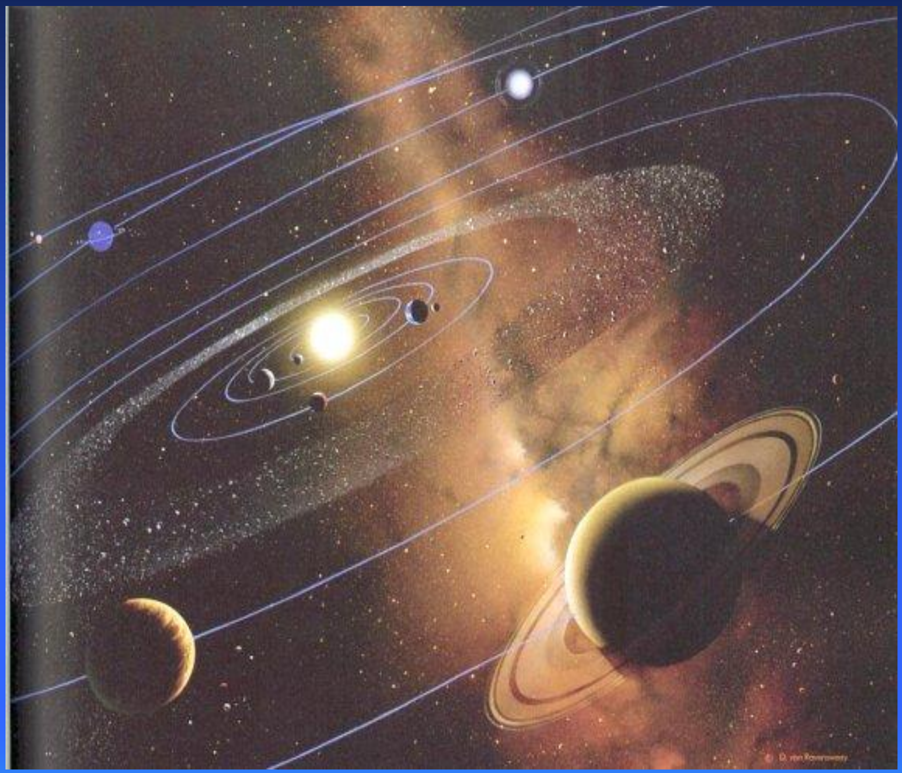
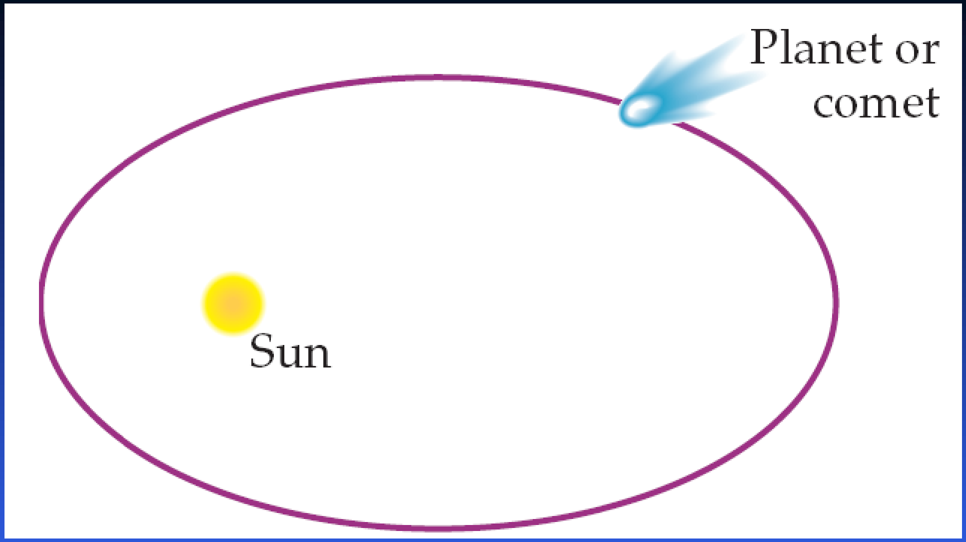
Principle Characteristics of the Planets

Name	*Distance from Sun (AU)*	*Revolution Period (Years)*	*Diameter (km)*	*Mass (10^{23} kg)*	*Density (g/cm^3)*
Mercury	0.39	0.24	4878	3.3	5.4
Venus	0.72	0.62	12102	48.7	5.3
Earth	1.00	1.00	12756	59.8	5.5
Mars	1.52	1.88	6787	6.4	3.9
Jupiter	5.20	11.86	142984	18991	1.3
Saturn	9.54	29.46	120536	5686	0.7
Uranus	19.18	84.07	51118	866	1.2
Neptune	30.06	164.82	49660	1030	1.6
Pluto	39.44	248.60	2200	0.01	2.1

Kepler's first law

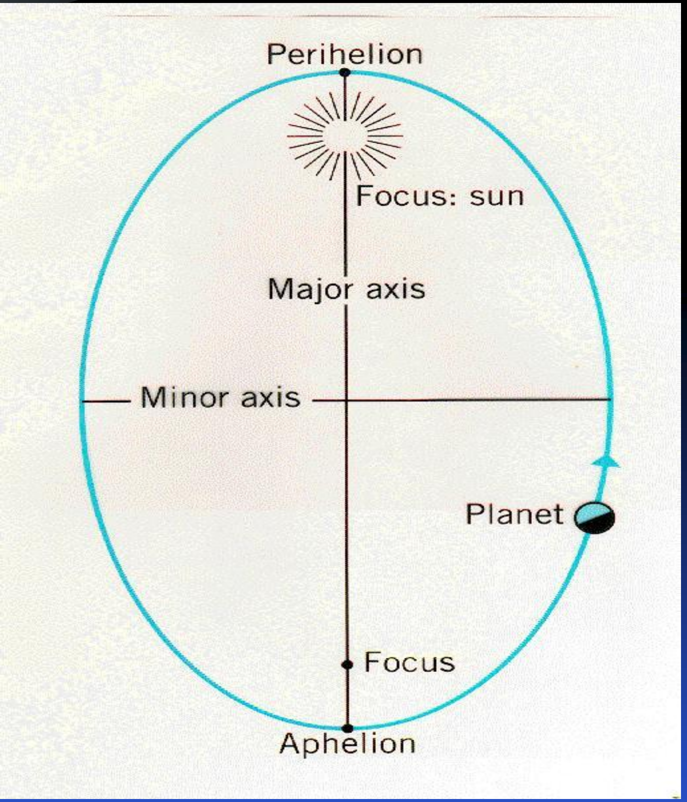
Planets follow elliptical orbits, with the Sun at one focus of the ellipse.





- planet's orbit the Sun in ellipses, with the Sun at one focus.
- the eccentricity of the ellipse, e, tells you how elongated it is.
- e=0 is a circle, e<1 for all ellipses

e=0.02 e=0.4 e=0.7



Orbit Examples

- Pluto has the highest eccentricity of any planet
  - $e_{\text{Pluto}} = 0.25$
- Halley's comet has an orbit with high eccentricity
  - $e_{\text{Halley's comet}} = 0.97$

eccentricity of the planets

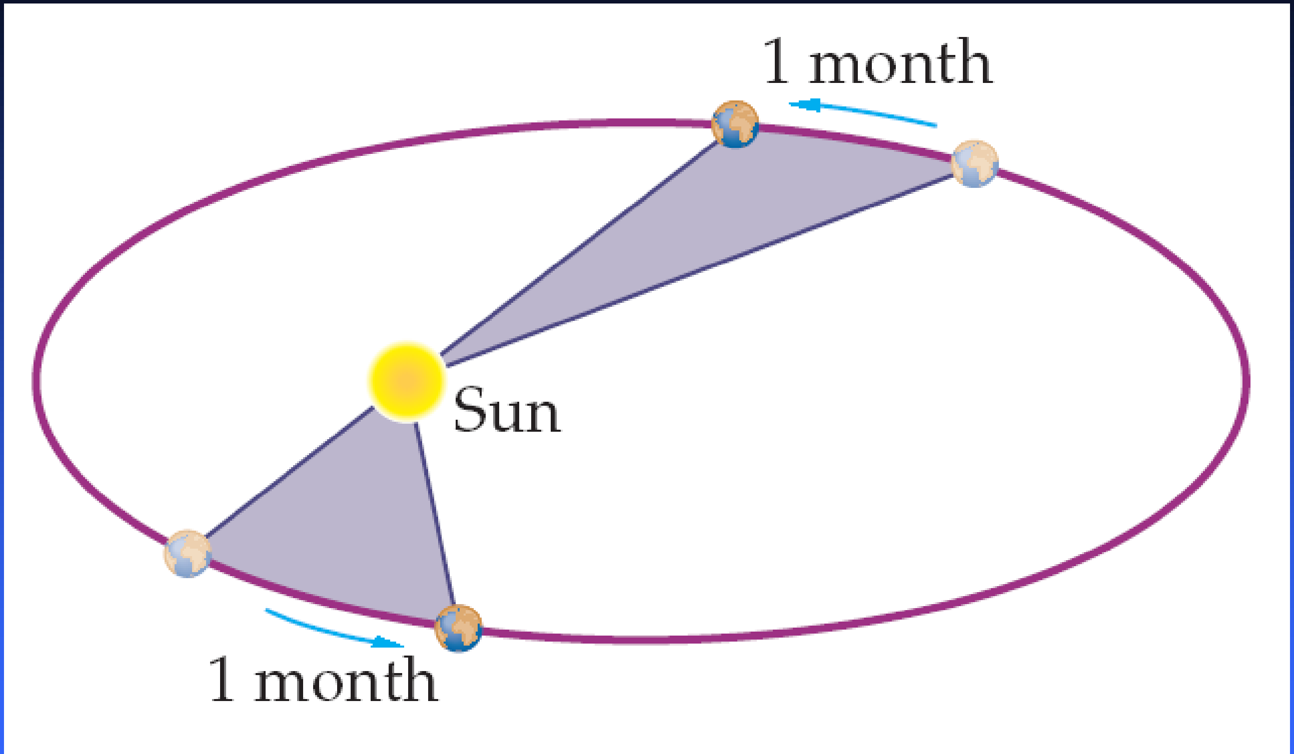
Mercury	0.206	Saturn	0.054
Venus	0.007	Uranus	0.048
Earth	0.017	Neptune	0.007
Mars	0.094	Pluto	0.253
Jupiter	0.048		

Notes About Ellipses Planet Orbits

- The Sun is at one focus
  - Nothing is located at the other focus
- **Aphelion** (afelion) is the point farthest away from the Sun
  - The distance for aphelion is $a + c$
    - For an orbit around the Earth, this point is called the apogee (apogee)
- **Perihelion** is the point nearest the Sun
  - The distance for perihelion is $a - c$
    - For an orbit around the Earth, this point is called the perigee (perigee)

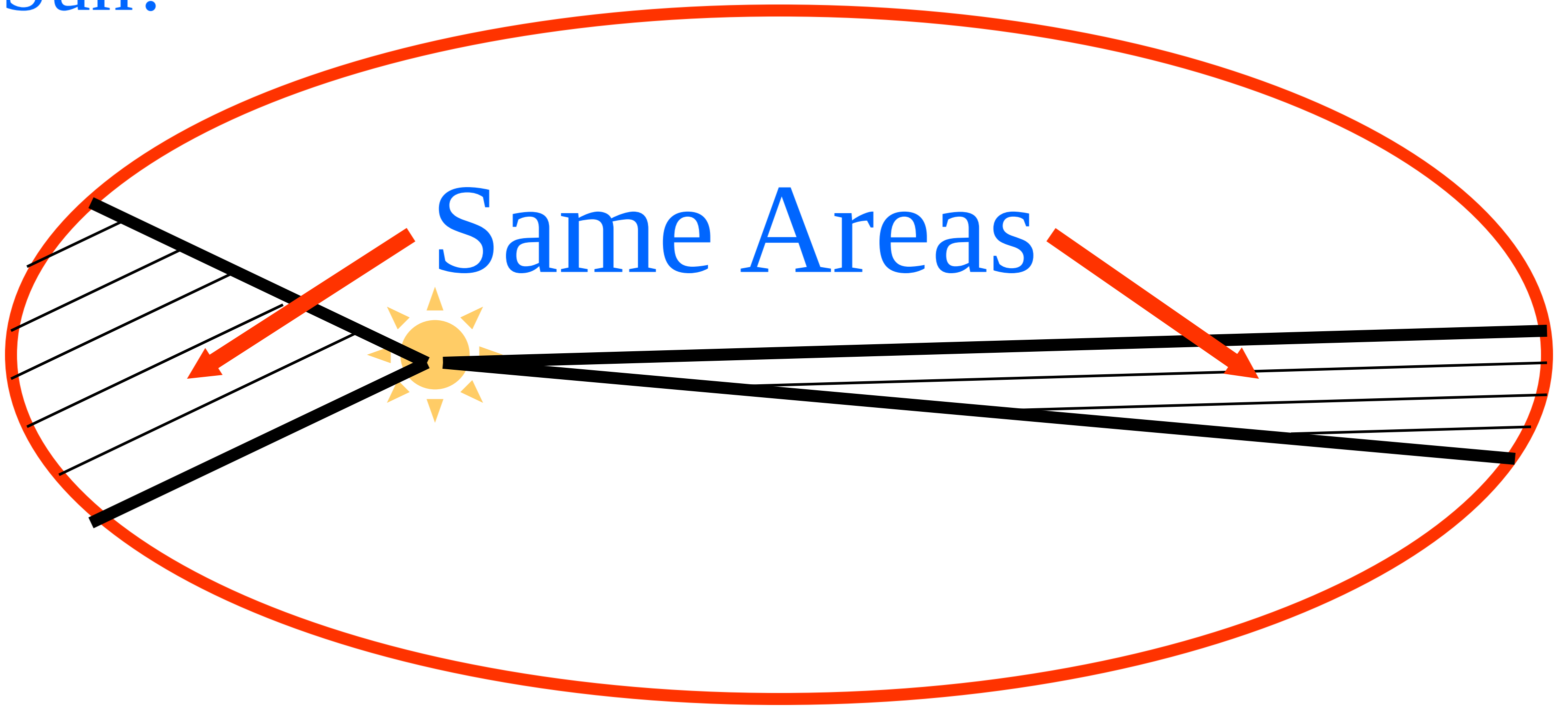
Kepler’s second law

The line joining the Sun and a planet sweeps out equal areas in equal time intervals.



As a result, planets move fastest when they are near the Sun (perihelion) and slowest when they are far from the Sun (aphelion).

Q. If the planet sweeps out equal areas in equal times, does it travel faster or slower when far from the Sun?



Law of Areas

A line that connects a planet to the sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals; that is the rate dA/dt at which it sweeps out area A is constant

Area ΔA of the wedge is the area of the triangle i.e.

$\Delta A=(r^2\Delta\theta)/2$;

$dA/dt = r^2(d\theta/dt)/2= r^2\omega/2$

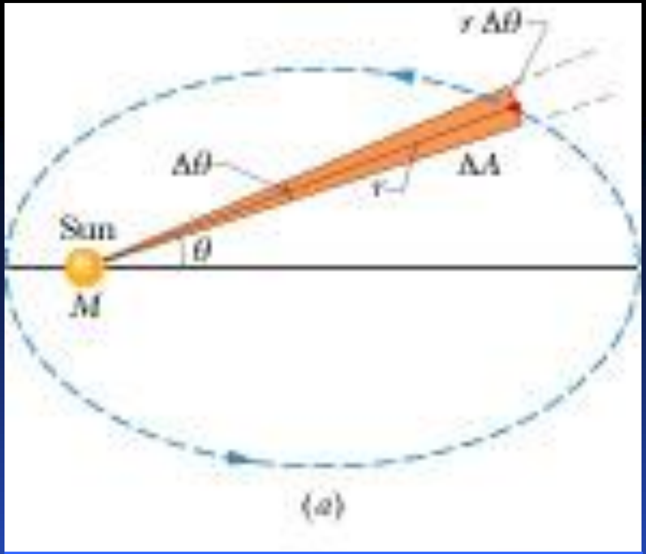
But angular momentum $L=mr^2\omega$

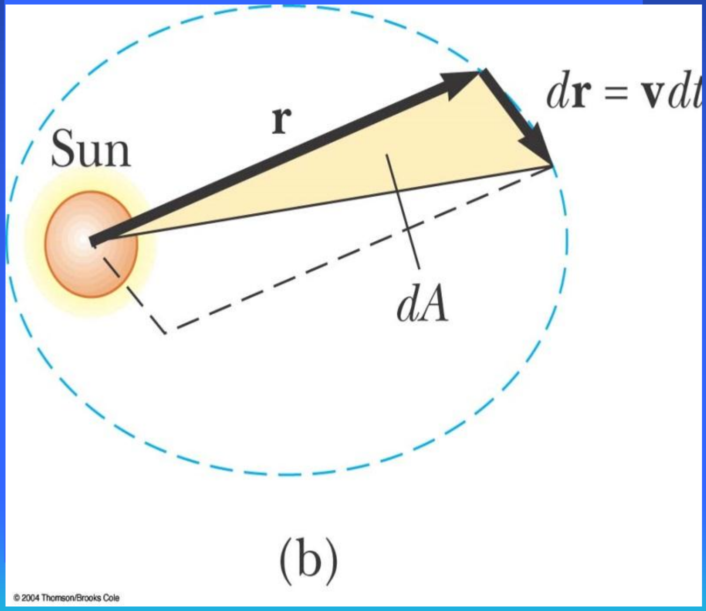
Then $dA/dt= r^2\omega/2=L/2m$

In central field L is constant because in central field: $\vec{F} = f(r)\hat{r}$

Or in Fig.(b):

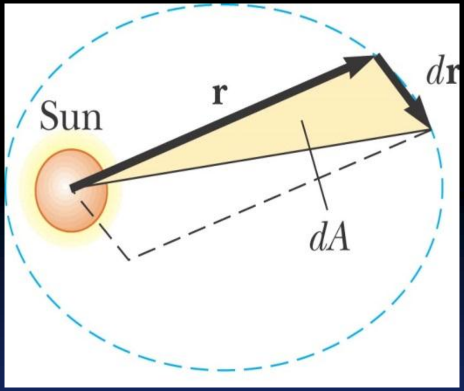
$dA=|\mathbf{r} \times d\mathbf{r}|/2=\ldots.=L/2m$





Example: Kepler's Second Law and Angular Momentum Conservation.

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *conservative central force field*. Therefore the torque acting on the planet by this force is always 0.

$$\vec{\tau} = \vec{r} \times \vec{F} = r\hat{r} \times f(r)\hat{r} = rf(r)[\hat{r} \times \hat{r}] = 0$$

Since torque is the time rate change of angular momentum *L*, the angular momentum is constant.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \begin{aligned} \vec{L} &= const. = \vec{r} \times \vec{p} \\ &= \vec{r} \times M_p \vec{v} = M_p (\vec{r} \times \vec{v}) \end{aligned}$$

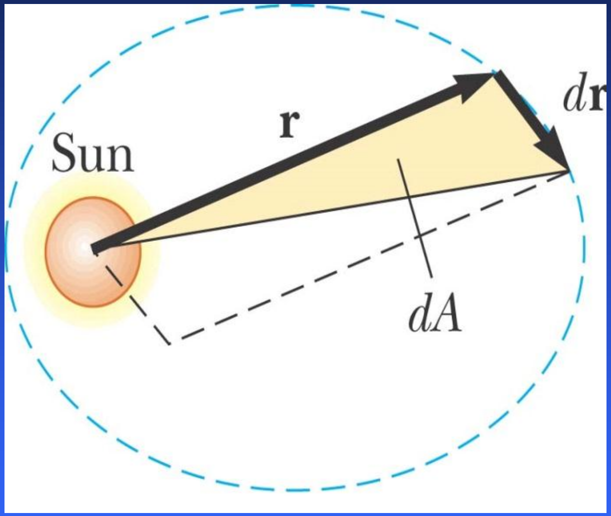
Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$L = const. = |\vec{r} \times \vec{p}| = |\vec{r} \times M_p \vec{v}|$$
$$= M_p |\vec{r} \times \vec{v}|$$

Since the area swept by the motion of the planet is:

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}|\vec{r} \times \vec{v}|dt$$

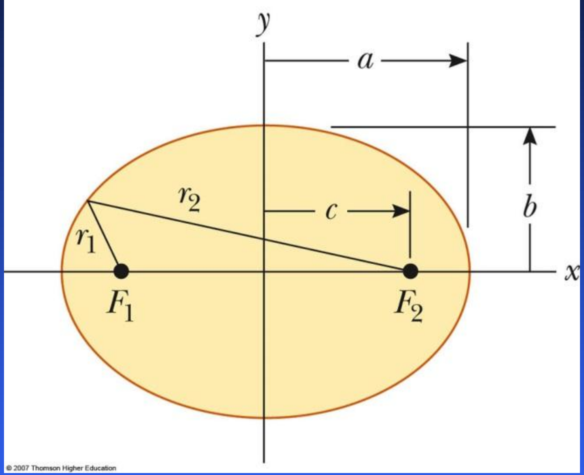
$$\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{v}| = \frac{L}{2M_p} = const.$$



This is Keper's second law which states that the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

Kepler's Third Law

- The square of the orbital period of any planet is proportional to cube of the average distance from the Sun to the planet.

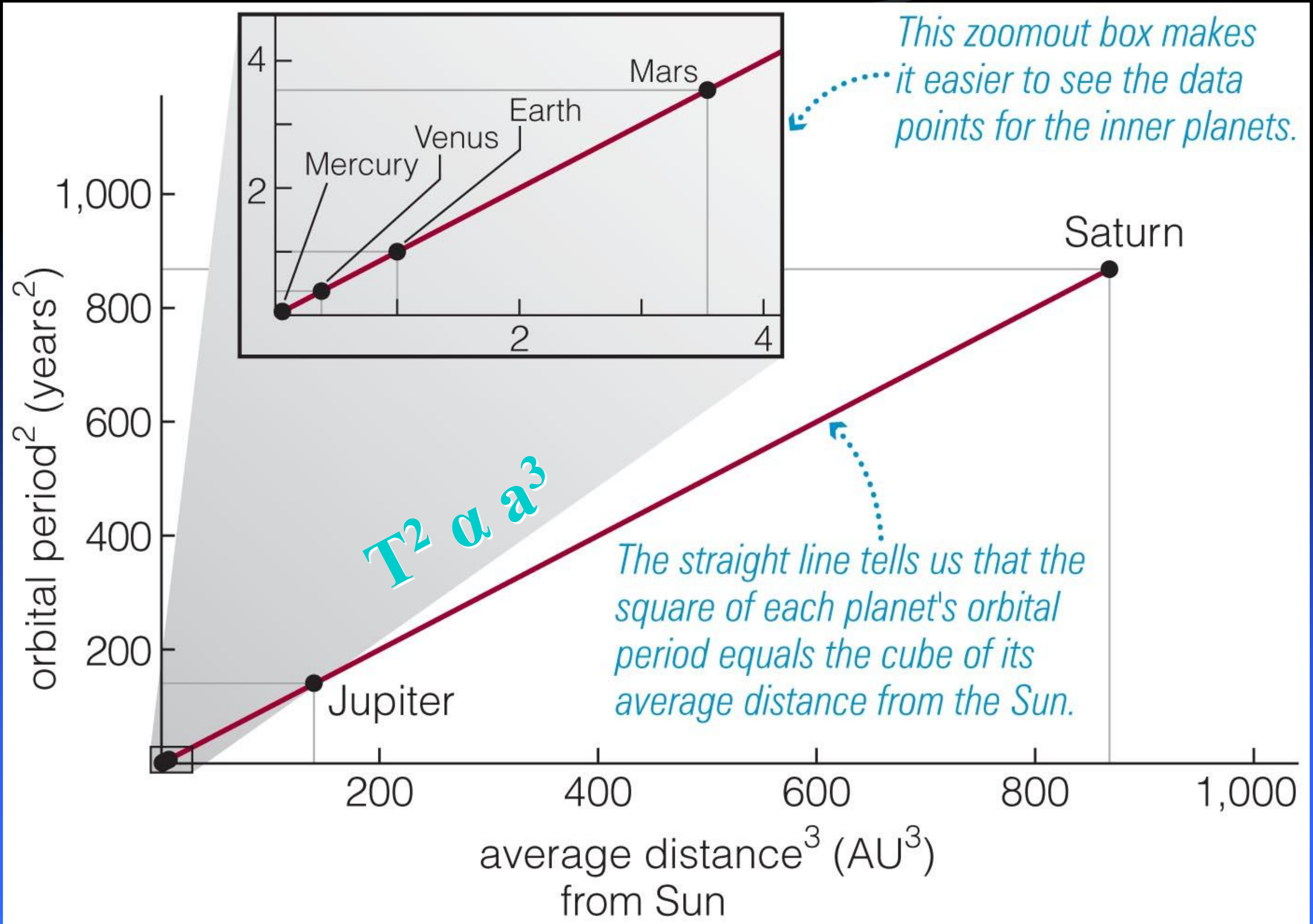


$$T^2 = K a^3$$

  - T is the period of the planet
  - a is the length of the semi-major axis
  - For orbit around the Sun, $K = K_S = 2.97 \times 10^{-19}\ s^2/m^3$
  - K is independent of the mass of the planet and K=1 in A.U.
  - 1 A.U. = 1.5×10^8 km

$$K_s = \frac{4\pi^2}{GM_s}$$

G is the gravitational constant = $6.673 \times 10^{-11}\ m^3\ kg^{-1}\ sec^{-2}$



Kepler's Third Law

Some Important Points of Kepler's Third Law

- Approximately, $T^2_{earth}/a^3_{earth} = T^2_{planet}/a^3_{planet}$
- Sometimes we use Earth-years and Earth-distance to the Sun (1 A.U.) as units.
- The constant of proportionality depends on the mass of the Sun--and that's how we know the mass of the Sun.
- We can apply this to moons (or any satellite) orbiting a planet, and then the constant of proportionality depends on the mass of the planet.
- Newton's law of gravitation, and Newton's second law (net force = mass x acceleration) can be used to derive Kepler's three laws of planetary motion.

Example: Derivation of Kepler's Third Law from Newton's law of gravitation, and Newton's second law

- Can be predicted from the inverse square law
- Start by assuming a circular orbit
- The gravitational force supplies a centripetal force
- K_s is a constant
- This can be extended to an elliptical orbit
- Replace r with a
  - Remember a is the semimajor axis

$$\frac{GM_{\text{Sun}}M_{\text{Planet}}}{r^2} = \frac{M_{\text{Planet}}v^2}{r}$$

$$v = \frac{2\pi r}{T}$$

$$T^2 = \left(\frac{4\pi^2}{GM_{\text{Sun}}}\right)r^3 = K_S r^3$$

$$T^2 = \left(\frac{4\pi^2}{GM_{\text{Sun}}}\right)a^3 = K_S a^3$$

- K_s is independent of the mass of the planet, and so is valid for any planet
- If an object is orbiting another object, the value of K will depend on the object being orbited
- For example, for the Moon around the Earth, K_{Sun} is replaced with K_{Earth}

Example

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.16×10^7s, and its distance from the Sun is 1.496×10^{11}m.

Using Kepler's third law.

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3 = K_s r^3$$

The mass of the Sun, M_s, is

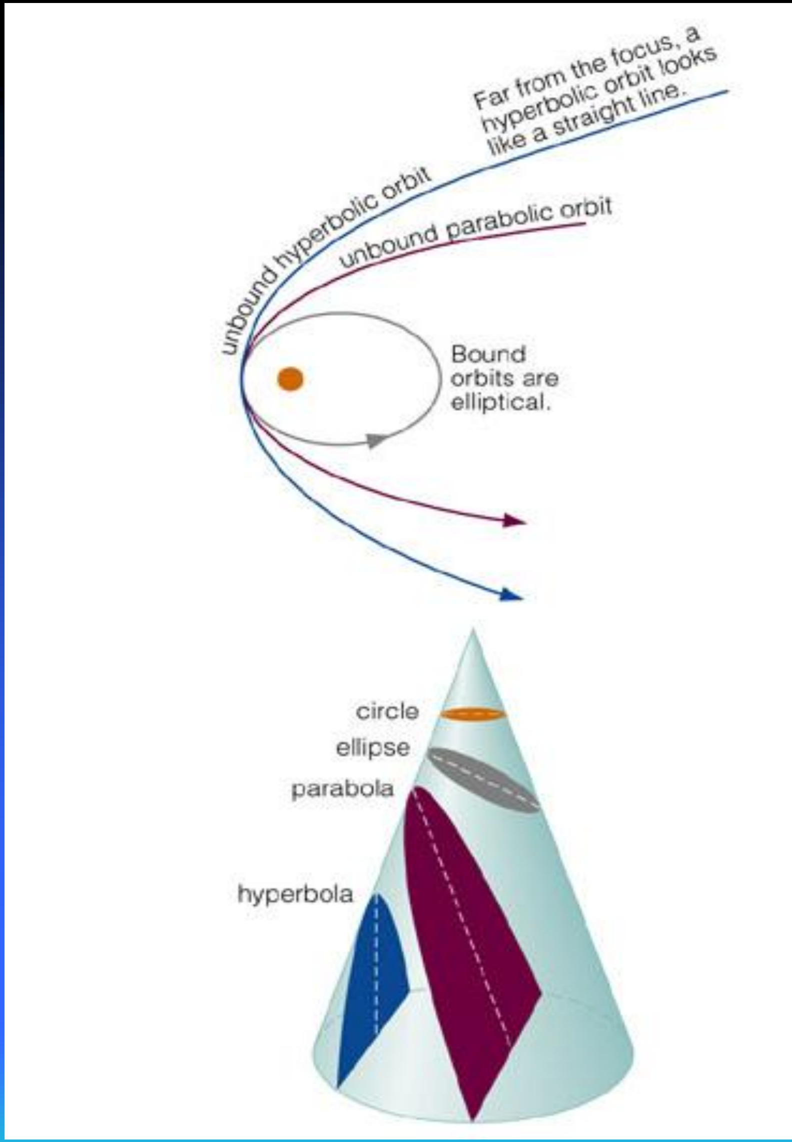
$$M_s = \left(\frac{4\pi^2}{GT^2}\right) r^3$$

$$= \left(\frac{4\pi^2}{6.67 \times 10^{-11} \times (3.16 \times 10^7)^2}\right) \times \left(1.496 \times 10^{11}\right)^3$$

$$= 1.99 \times 10^{30} kg$$

Orbital Paths

- Extending Kepler's Law, Newton found that ellipses were not the only orbital paths.
- possible orbital paths
  - circle (bound) – e=0
  - ellipse (bound) - e<1
  - parabola (unbound) – e=1
  - hyperbola (unbound) – e>1



Orbit of a particle in a central force field

$$m\vec{a} = \vec{F} = f(r)\hat{r}$$

$$m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}] = f(r)\hat{r}$$

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \quad ...(1)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad ...(2)$$

$$Eq.(2) * r \rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$r^2\dot{\theta} = const. = \frac{L}{m} = h$$

Angular momentum per unit mass
(Keper's second law)

To find the equation of the orbit from Eq.(1), we shall use the variable u defined by:

$$u = \frac{1}{r} \rightarrow r = \frac{1}{u}$$

$$h = r^2\dot{\theta} = \frac{\dot{\theta}}{u^2} \rightarrow \dot{\theta} = hu^2$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{d\theta}{dt}\frac{du}{d\theta} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

$$\ddot{r} = -h\frac{d}{dt}(\frac{du}{d\theta}) = -h\frac{d}{d\theta}(\frac{du}{d\theta})\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Using the above variables Eq.(1) becomes:

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$-mh^2u^2\frac{d^2u}{d\theta^2} - m\frac{1}{u}h^2u^4 = f(\frac{1}{u}) = f(u^{-1})$$

Thus, the differentia equation of the orbit of a particle moving under a central field is:

$$\left(\frac{d^2u}{d\theta^2}+u\right)=-\frac{f(u^{-1})}{mh^2u^2}$$

$*\ If\ we\ have\ \ f(u^{-1})\rightarrow we\ obtain\ \ u=u(\theta)$

$*\ from\ u=u(\theta)\rightarrow we\ obtain\ \ f(u^{-1})$

Energy Equation of the orbit

In a central force field the force is a function of r only and it is conservative. Thus,

$$T + V = const. = E \rightarrow \frac{1}{2}mv^2 + V(r) = E$$

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

using

$$r = \frac{1}{u},\ \dot{\theta} = hu^2 \ \&\ \dot{r} = -h\frac{du}{d\theta}$$

the energy equation of the orbit of a particle moving under a central field is:

$$\frac{1}{2}mh^2[(\frac{du}{d\theta})^2 + u^2) + V(u^{-1}) = E$$

Ex.: A particle in a central field moves in the spiral orbit $r = c\Theta^2$.

(a) Determine the form of the force function using differential equation of the orbit.

(b) Determine the form of the force function using energy equation of the orbit.

(c) Determine how the angle Θ varies with the time.

(a)

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{f(u^{-1})}{mh^2u^2}$$

$$u = \frac{1}{r} = \frac{1}{c\theta^2} \rightarrow \frac{du}{d\theta} = \frac{-2}{c}\theta^{-3} \rightarrow \frac{d^2u}{d\theta^2} = \frac{6}{c}\theta^{-4} = 6cu^2$$

$$6cu^2 + u = \frac{-f(u^{-1})}{mh^2u^2} \rightarrow f(u^{-1}) = -mh^2(6cu^4 + u^3)$$

$$\therefore f(r) = -mh^2\left(\frac{6c}{r^4} + \frac{1}{r^3}\right)$$

(b)

$$\frac{1}{2}mh^2[(\frac{du}{d\theta})^2+u^2)+V(u^{-1})=E$$

$$u=\frac{1}{r}=\frac{1}{c\theta^2}\rightarrow\frac{du}{d\theta}=\frac{-2}{c}(\sqrt{cu})^3=-2c^{1/2}u^{3/2}$$

$$\frac{1}{2}mh^2[4cu^3+u^2)+V(u^{-1})=E$$

$$V(r)=E-\frac{1}{2}mh^2[\frac{4c}{r^3}+\frac{1}{r^2})$$

$$f(r)=-\frac{\partial V(r)}{\partial r}=-[0-\frac{1}{2}mh^2[\frac{-12c}{r^4}-\frac{2}{r^3})]$$

$$\therefore\ f(r)=-mh^2(\frac{6c}{r^4}+\frac{1}{r^3})$$

(c)

$$\dot{\theta} = hu^2 = \frac{h}{c^2\theta^4} = \frac{d\theta}{dt}$$

$$\int_0^\theta \theta^4 d\theta = \int_0^t \frac{h}{c^2} dt \rightarrow \frac{\theta^5}{5} = \frac{h}{c^2} t$$

$$\theta = (5hc^{-2})^{1/5} t^{1/5} \rightarrow \theta \propto t^{1/5}$$

Ex.: A particle in a central field moves in a logarithmic spiral orbit given by, $r=k\, e^{\alpha\,\Theta}$ where k and α are constants.

(a) Determine the form of the force function using differential equation of the orbit.

(b) Determine the form of the force function using energy equation of the orbit.

(c) Determine how the angle Θ varies with the time.

Ex. (H.W.)

(a) Orbits of a particle in an inverse square field.

(b) Orbital energies of a particle in the inverse square field.

$$(\frac{d^2u}{d\theta^2}+u)=-\frac{f(u^{-1})}{mh^2u^2}$$

$$f(r)=-\frac{k}{r^2}\rightarrow f(u^{-1})=-ku^2$$

$$\frac{1}{2}mh^2[(\frac{du}{d\theta})^2+u^2)+V(u^{-1})=E$$

$$V(r)=-\frac{k}{r}\rightarrow V(u^{-1})=-ku$$

End of the Lecture

Let Learning Continue

Thank You