Chapter Three

Fluid in Motion

3.1 Introduction:

- The **path of an individual particle in a moving fluid** is called a **flow line**.
- **If the overall flow pattern does not change with time**; the flow is called **steady flow**.
- **The flow lines passing through the edge of an imaginary element of area,** such as the area (A) in Fig. [1]**, form a tube called a flow tube**.

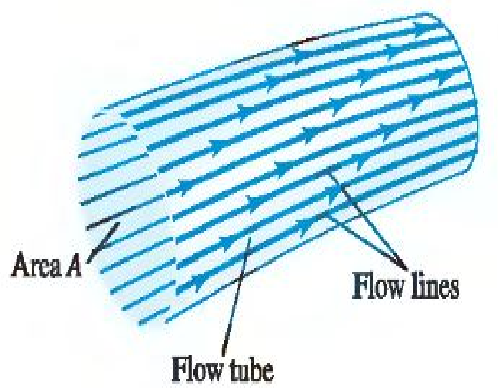


Figure [1]: A flow tube hounded by flow lines. In steady flow, fluid cannot cross the walls of a flow tube.

- This is called turbulent flow (Fig. 2). In **turbulent flow there is no steady-state pattern; the flow pattern changes continuously**.

Figure [2]: Theflowofsmokerisingfrom these incense sticks is laminar up to a certain point, and then becomes turbulent.

3.2 Equation of Continuity of Flow:

The rate of flow of fluid into a system equals the rate of flow out of the system.

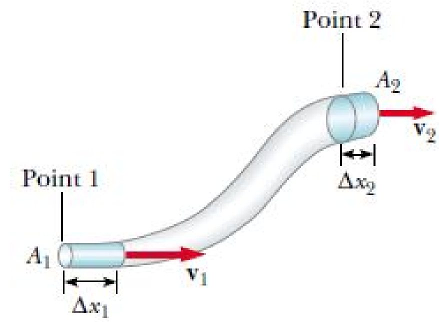


Figure [3]: A flow tube with changing cross-sectional area. If the fluid is incompressible, the product Av has the same value at all points along the tube.

Let's first consider the case of an incompressible fluid so that the **density ρ** has the same value at all points. The **mass dm_1** flowing into the tube across A_1 in **time dt** is: $dm_1 = \rho A_1 v_1 dt$. Similarly, the **mass dm_2** that flows out across A_2 in the same **time** is: $dm_2 = \rho A_2 v_2 dt$, as shown in figure [3]. **In steady flow the total mass in the tube is constant, so**:

$$dm_1 = dm_2$$

$$d(\rho V_1) = d(\rho V_2)$$

$$\rho d(A_1 x_1) = \rho d(A_2 x_2)$$

$$\rho A_1 d(v_1 t) = \rho A_2 d(v_2 t)$$

$$\rho A_1 v_1 dt = \rho A_2 v_2 dt$$

$$A_1 v_1 = A_2 v_2 = constant \quad = \quad \emptyset \quad \ldots (1)$$

where **v is the fluid speed**, and **x is the distance**.

This expression is called the **equation of continuity for fluids**.

The **product of the area and the fluid speed at all points along a pipe (Av)** is the **volume flow rate** (φ), the rate at which volume crosses a section of the tube:

$$\varphi = \frac{dV}{dt} = Av \quad \ldots (2)$$

The **mass flow rate** **is the mass flow per unit time** through a cross section. This is equal to the **density ρ** times the **volume flow rate dV/dt**. That is: $\because\ \rho = \frac{m}{V} \rightarrow V = \frac{m}{\rho}$

$$\frac{dm}{dt} = \rho A v \quad \dots (3)$$

We can generalize Eq. (1) for the case in which the fluid is **not incompressible**.

If ρ_1 and ρ_2 are the densities at sections 1 and 2, then:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \quad \dots (4)$$

If the fluid is denser at point 2 than at point 1 ($\rho_2 > \rho_1$), the **volume flow rate** at point 2 will be **less than** at point 1 ($A_2 v_2 < A_1 v_1$).

3.3 Rate of flow:

The rate of a liquid is defined as the volume of it that flows across any section in unit time.

Let **v : velocity of liquid**.

A: area.

L: distance between two sections.

t : time taken by the liquid to flow L distance.

$$v = \frac{L}{t} \Rightarrow L = vt$$

Volume of the liquid flowing through the section OQ: $V = LA = vtA$

∴ Ratio of flow of liquid:

$$\varphi = \frac{dV}{dt} = \frac{v\cancel{t}A}{\cancel{t}} = Av$$

3.4 Energy of the Fluid:

We have **three types of the energy** possessed by a liquid in flow:

1- Kinetic Energy (KE):

$$KE = \frac{1}{2}mv^2$$

$$\boxed{\rho = \frac{m}{V}}$$

$$KE\ per\ unit\ volume = \frac{1}{2}\rho v^2$$

2- Potential Energy (PE):

$$\boxed{\rho = \frac{m}{V}}$$

$$PE = mgh$$

$$PE\ per\ unit\ volume = \rho gh$$

3- Pressure Energy (Energy = Work):

***P*: Pressure of the liquid**.

$$\boxed{P = \frac{F}{A}}$$

$$Pressure\ Energy = FL = PAL = PV$$

$$Pressure\ Energy\ per\ unit\ volume = P$$

The three types of energy possessed by a liquid under flow are **mutually convertible**, one into the other, and **this sum will be constant**.

3.5 Bernoulli's Equation:

Bernoulli's equation **is not** a free-standing law of physics; rather, **it's a consequence of energy conservation as applied to an ideal fluid**.

Daniel Bernoulli (1700–1782):

Daniel Bernoulli, a Swiss physicist and mathematician, made important discoveries in fluid dynamics.

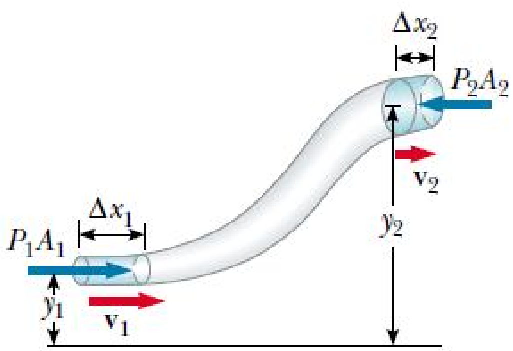


Figure [4]: A fluid in laminar flow through a constricted pipe. The volume of the shaded section on the left is equal to the volume of the shaded section on the right.

The **relationship between fluid pressure (P), speed (v), and elevation (y)** was **first derived in 1738** by the **Swiss physicist Daniel Bernoulli**. Consider the **flow of an ideal fluid** through a no uniform pipe in a time t, as illustrated in Figure (4).

Let us call the **lower** shaded part **section 1** and the **upper** shaded part **section 2**.

The **work done by the fluid in section 1 (W_1)** in a **time t** is:

$$W_1 = F_1 \Delta x_1 = P_1 A_1 \Delta x_1 = P_1 V \quad \dots (5)$$

Where the **force exerted by the fluid in section 1 (F_1)** has a magnitude $F_1 = P_1 A_1$.

where **V is the volume of section 1**.

In a similar manner, the **work done by the fluid in section 2 (W_2)** in the same **time t** is:

$$W_2 = -F_2 \Delta x_2 = -P_2 A_2 \Delta x_2 = -P_2 V \quad \dots (6)$$

(The volume that passes through section 1 in a **time t** equals the volume that passes through section 2 in the same **time**.) This **work is negative because the fluid force opposes the displacement**.

Thus, the **net work done (W)** by these forces in the **time t** is:

$$W = (P_1 - P_2)V \quad \dots(7)$$

Part of this work goes into changing the kinetic energy of the fluid, and part goes into changing the gravitational potential energy. If **m is the mass** that enters one end and leaves the other in a **time t**, then the **change in the kinetic energy (ΔK)** of this mass is:

$$\Delta K = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m{v_1}^2 \quad \dots(8)$$

The **change in gravitational potential energy (ΔU)** is:

$$\Delta U = mgy_2 - mgy_1 \quad \dots(9)$$

Therefore, the **total work done** is:

$$W = \Delta K + \Delta U \quad \dots(10)$$

Substituting equations (7), (8) and (9) into equation (10), we get:

$$(P_1 - P_2)V = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m{v_1}^2 + mgy_2 - mgy_1$$

If we **divide each term by V** and recall that $\rho = m/V$, this expression reduces to:

$$P_1 - P_2 = \frac{1}{2}\rho{v_2}^2 - \frac{1}{2}\rho{v_1}^2 + \rho gy_2 - \rho gy_1$$

$$P_1 + \frac{1}{2}\rho{v_1}^2 + \rho gy_1 = P_2 + \frac{1}{2}\rho{v_2}^2 + \rho gy_2 \quad \dots(11)$$

This is **Bernoulli's equation** as applied to an ideal fluid. It is often expressed as:

$$P + \frac{1}{2}\rho v^2 + \rho gy = Constant \quad \dots(12)$$

Bernoulli's equation states that **the sum of the pressure P, the kinetic energy per unit volume $\frac{1}{2}\rho v^2$, and the potential energy per unit volume ρgy, has the same value at all points along a streamline**.

3.6 Bernoulli's Principle for Gases (Venturi meter):

An important consequence of Bernoulli's equation can be demonstrated by considering Figure [5], which shows **water flowing through a horizontal constricted pipe from a region of large cross-sectional area into a region of smaller cross-sectional area**. This device, called a **Venturi tube**, can be **used to** measure the speed of fluid flow. **Because the pipe is horizontal $y_1 = y_2$**, and equation (11) applied to points 1 and 2 gives:

$$P_1 + \frac{1}{2}\rho {v_1}^2 = P_2 + \frac{1}{2}\rho {v_2}^2 \quad ...(13)$$

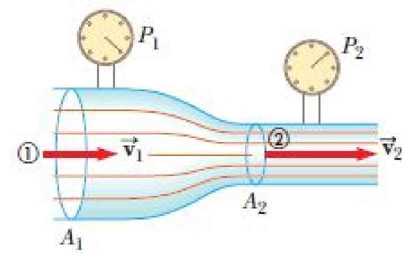


Figure [5]

In figure [5], **the pressure P_1 is greater than the pressure P_2 (i.e., $P_1 > P_2$), because $v_1 < v_2$**. This device can be **used to** measure the speed of fluid flow. This result is often expressed by the statement that **swiftly moving fluids** exert **less pressure than** do **slowly moving fluids**.

3.7 Torricelli's Law:

The velocity of efflux of a liquid through an orifice is equal to that which a body attains in falling freely the surface of the liquid to the orifice.

- $Total\ Energy = \boldsymbol{KE} + \boldsymbol{PE} + \cancel{\boldsymbol{Pressure\ Energy}}$
- Total Energy at Point (1) = $0 + \boldsymbol{\rho g h} + 0$
- Total Energy at Point (2) = $\boldsymbol{\frac{1}{2}\rho v^2} + 0 + 0$

Since total energy remains the same:

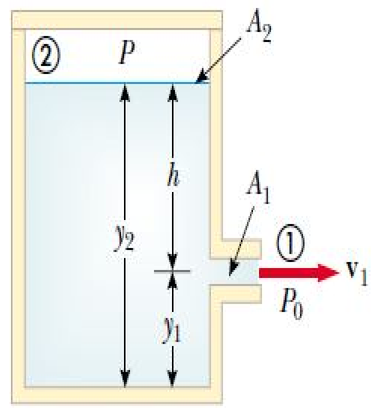
$$\therefore \cancel{\rho} g h = \frac{1}{2}\cancel{\rho} v^2$$

$$\therefore v^2 = 2gh$$

$$v = \sqrt{2gh}$$

Prove it!

For example: An enclosed tank containing a liquid of **density** ρ has a hole in its side at a **distance** y_1 from the tank's bottom (this Figure). The **hole** is open to the atmosphere, and its **diameter is much smaller than the diameter of the tank**. The **air above the liquid** is maintained at a **pressure** P. Determine the **speed** of the **liquid** as it **leaves** the **hole** when the liquid's level is a **distance** h above the hole.



A liquid leaves a hole in a tank at speed v_1.

Solution:

Because $A_2 \gg A_1$, the liquid is approximately at rest at the top of the tank, where the pressure is P. **Applying Bernoulli's equation** to points 1 and 2 and noting that at the hole **P_1 is equal to atmospheric pressure P_0**, we find that:

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P + \rho g y_2$$

But $y_2 - y_1 = h$; thus, this expression reduces to:

$$v_1 = \sqrt{\frac{2(P - P_0)}{\rho} + 2gh}$$

When **P is much greater than P_0** (so that the term $2gh$ can be ***neglected***), the exit speed of the water is mainly a function of P.

$$v_1 = \sqrt{\frac{2(P - P_0)}{\rho}}$$

If the **tank is open to the atmosphere**, then:

$$P = P_0 \text{ and } v_1 = \sqrt{2gh}$$

Example 1: As part of a lubricating system for heavy machinery, oil of density $\mathbf{850\ kg/m^3}$ is pumped through a cylindrical pipe of **diameter** $\mathbf{8.0\ cm}$ at a **rate** of $\mathbf{9.5\ liters\ per\ second}$.

(a) What is the **speed of the oil**? What is the **mass flow rate**?

(b) If the pipe **diameter** is reduced to $\mathbf{4.0\ cm}$, what are the new values of the **speed** and **volume flow rate**? Assume that the **oil** is incompressible.

$$A = \pi r^2 = \frac{\pi}{4} D^2$$

Solution:

$$\frac{dV}{dt} = Av$$

$$v_1 = \frac{dV/dt}{A_1} = \frac{(9.5\ \mathrm{L/s})(10^{-3}\ \mathrm{m^3/L})}{\pi(4.0 \times 10^{-2}\ \mathrm{m})^2} = 1.9\ \mathrm{m/s}$$

The mass flow rate is

$$\frac{dm}{dt} = \rho A v = \rho\, dV/dt = (850\ \mathrm{kg/m^3})(9.5 \times 10^{-3}\ \mathrm{m^3/s}) = 8.1\ \mathrm{kg/s}.$$

$$A_1 v_1 = A_2 v_2$$

$$v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi(4.0 \times 10^{-2}\ \mathrm{m})^2}{\pi(2.0 \times 10^{-2}\ \mathrm{m})^2}(1.9\ \mathrm{m/s}) = 7.6\ \mathrm{m/s}$$

$$\frac{dV_2}{dt} = A_2 v_2 = \pi(2.0 \times 10^{-2})^2 \times 7.6 = 0.00955\ m^3/s$$

Example 2: **Each second**, $\mathbf{5525\ m^3}$ of **water** flows over the $\mathbf{670\ m}$ **wide** cliff of the Horseshoe Falls portion of **Niagara Falls**. The water is approximately $\mathbf{2\ m}$ **deep** as it reaches the cliff. What is its **speed** at that instant?

Solution:

$$A = (670\ \mathrm{m})(2\ \mathrm{m}) = 1\,340\ \mathrm{m^2}.$$

$$\frac{dV}{dt} = Av$$

$$v = \frac{5\,525\ \mathrm{m^3/s}}{A} = \frac{5\,525\ \mathrm{m^3/s}}{1\,340\ \mathrm{m^2}} = \boxed{4\ \mathrm{m/s}}$$

Example 3: **Water enters** a house through a pipe with an inside **diameter** of $\mathbf{2.0}$ $\boldsymbol{cm}$ at an **absolute pressure** of $\mathbf{4 \times 10^5}$ $\boldsymbol{Pa}$ $(\boldsymbol{about\ 4\ atm})$. A $\mathbf{1.0}$ $\boldsymbol{cm}$ **diameter** pipe leads to the **second-floor** bathroom $\mathbf{5.0}$ $\boldsymbol{m}$ **above**. When the flow **speed** at the **inlet pipe** is $\mathbf{1.5}$ $\boldsymbol{m/s}$, find the flow **speed**, **pressure**, and **volume flow rate** in the bathroom.

$$\boldsymbol{A = \pi r^2 or\ A = \frac{\pi}{4}D^2, y_2 - y_1 = h}$$

Solution:

$$\boldsymbol{d_1 = 2.0\ cm \rightarrow r_1 = 1.0\ cm\ \ and\ \ d_2 = 1.0\ cm \rightarrow r_2 = 0.5\ cm}$$

$$v_2 = \frac{A_1}{A_2}v_1 = \frac{\pi(1.0\text{ cm})^2}{\pi(0.50\text{ cm})^2}(1.5\text{ m/s}) = 6.0\text{ m/s}$$

We are given p_1 and v_1, and we can find p_2 from Bernoulli's equation:

$$p_2 = p_1 - \frac{1}{2}\rho(v_2^2 - v_1^2) - \rho g(y_2 - y_1) = 4.0 \times 10^5\text{ Pa}$$

$$-\frac{1}{2}(1.0 \times 10^3\text{ kg/m}^3)(36\text{ m}^2/\text{s}^2 - 2.25\text{ m}^2/\text{s}^2)$$

$$-(1.0 \times 10^3\text{ kg/m}^3)(9.8\text{ m/s}^2)(5.0\text{ m})$$

$$= 4.0 \times 10^5\text{ Pa} - 0.17 \times 10^5\text{ Pa} - 0.49 \times 10^5\text{ Pa}$$

$$= 3.3 \times 10^5\text{ Pa} = 3.3\text{ atm}$$

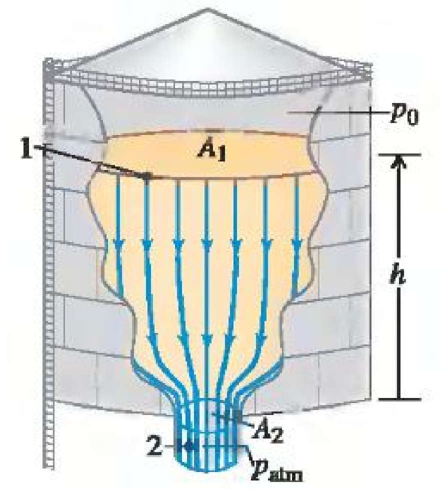
The volume flow rate is:

$$\frac{dV}{dt} = A_2 v_2 = \pi(0.50 \times 10^{-2}\text{ m})^2(6.0\text{ m/s})$$

$$= 4.7 \times 10^{-4}\text{ m}^3/\text{s} = 0.47\text{ L/s}$$

Example 4: This figure shows a gasoline storage tank with cross-sectional **area** $\boldsymbol{A_1}$, filled to a **depth** $\boldsymbol{h}$. The space above the gasoline contains **air** at **pressure** $\boldsymbol{P_0}$, and the gasoline flows out through a short pipe with **area** $\boldsymbol{A_2}$. Derive expressions for the flow **speed in the pipe** and the **volume flow rate**.

Solution:



We apply Bernoulli's equation to points 1 and 2

$$p_0 + \frac{1}{2}\rho v_1^2 + \rho g h = p_{atm} + \frac{1}{2}\rho v_2^2 + \rho g(0) \quad \boxed{\times \frac{2}{\rho}}$$

$$v_2^2 = v_1^2 + 2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh$$

Using $v_1 = 0$, we find

$$v_2^2 = 2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh$$

the volume flow rate is $dV/dt = v_2 A_2$.

If the top of the tank is vented to the atmosphere, $p_0 = p_{atm}$

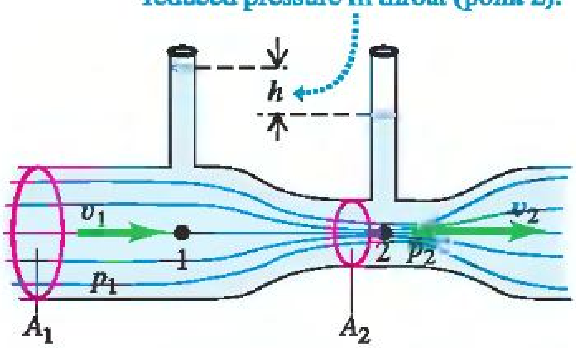
and there is zero pressure difference: $p_0 - p_{atm} = 0$.

In that case, $v_2 = \sqrt{2gh}$

In this case the volume flow rate is:

$$\frac{dV}{dt} = A_2\sqrt{2gh}$$

Example 5: This figure shows a **Venturi meter**, used to measure flow speed in a pipe. The narrow part of the pipe is called the throat. Derive an expression for the **flow speed** v, in terms of the **cross-sectional areas A_1 and A_2** and the difference in **height h** of the liquid levels in the **two vertical tubes**.



Solution:

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

From the continuity equation, $v_2 = (A_1/A_2)v_1$.

Substituting this and rearranging, we get

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2\left(\frac{A_1^2}{A_2^2} - 1\right)$$

the pressure difference $p_1 - p_2$ is also equal to $\rho g h$,

where h is the difference in the liquid levels in the two tubes.

Combining this with the above result and solving for v_1, we get

$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}$$

Example 6: If the **diameters** of a pip are $0.02m$ and $0.04m$. When a liquid of **density** $800Kg/m^3$ flows through it, the **difference in pressure between two positions** is $0.08m$. Calculate the **rate of flow** of the liquid through the tube.

Solution:

$$P_2 - P_1 = 0.08\ m = 800 \times 9.8 \times 0.08$$

$$A_1 = \pi R_1^2 = \pi\left(\frac{0.02}{2}\right)^2 = \pi(0.01)^2$$

$$A_2 = \pi R_2^2 = \pi\left(\frac{0.04}{2}\right)^2 = \pi(0.02)^2$$

$$P_1 + \frac{1}{2}\rho {v_1}^2 + \rho \cancel{g} h_1 = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho \cancel{g} h_2$$

For **horizontal** case:

$$P_1 + \frac{1}{2}\rho {v_1}^2 = P_2 + \frac{1}{2}\rho {v_2}^2$$

$$\frac{1}{2}\rho({v_1}^2 - {v_2}^2) = P_2 - P_1$$

$${v_1}^2 - {v_2}^2 = \frac{2(P_2 - P_1)}{\rho}$$

$${v_1}^2 = \frac{2(P_2 - P_1)}{\rho} + {v_2}^2$$

But: $v_1 A_1 = v_2 A_2 \rightarrow v_1 = \frac{A_2}{A_1} v_2$

Squaring both sides: $\frac{{A_2}^2}{{A_1}^2}{v_2}^2 = \frac{2(P_2 - P_1)}{\rho} + {v_2}^2 \rightarrow {A_2}^2{v_2}^2 = \frac{2(P_2 - P_1)}{\rho}{A_1}^2 + {A_1}^2{v_2}^2$

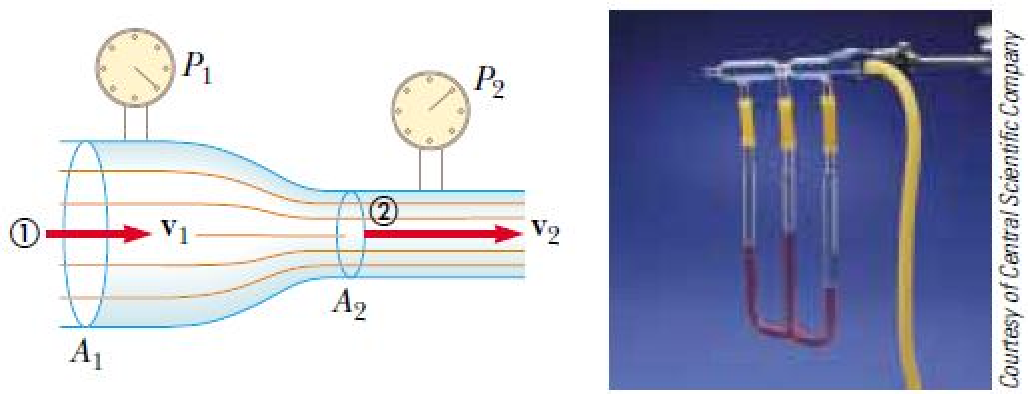
$${A_2}^2{v_2}^2 - {A_1}^2{v_2}^2 = \frac{2(P_2 - P_1)}{\rho}{A_1}^2 \rightarrow {v_2}^2 = \frac{2(P_2 - P_1)}{\rho\left({A_2}^2 - {A_1}^2\right)}{A_1}^2 \Rightarrow \boldsymbol{v_2 = A_1\sqrt{\frac{2(P_2 - P_1)}{\rho\left({A_2}^2 - {A_1}^2\right)}}}$$

$$\varphi = A_2 \boldsymbol{v_2} = A_1 A_2 \sqrt{\frac{2(P_2 - P_1)}{\rho\left({A_2}^2 - {A_1}^2\right)}}$$

$$\varphi = \pi^2(0.01)^2(0.02)^2\sqrt{\frac{2(800 \times 9.8 \times 0.08)}{800 \times \pi^2((0.02)^4 - (0.01)^4)}} \rightarrow \varphi = 4.062 \times 10^{-4}\, m^3/sec$$

Home Work:

Q 1: The horizontal constricted pipe illustrated in this figure, known as a Venturi tube, can be used to measure the flow speed of an incompressible fluid. Determine the flow speed at point 2 if the pressure difference $P_1 - P_2$ is known.



(a) (b)

Solution:

Q 2: An airplane has wings, each with area 4.00 m^2, designed so that air flows over the top of the wing at 245 m/s and underneath the wing at 222 m/s. Find the mass of the airplane such that the lift on the plane will support its weight, assuming the force from the pressure difference across the wings is directed straight upwards.

Solution:

Q 3: If the diameters of a pipe are 6 cm and 10 cm, at the points where a venturi meter is connected and the pressures at the points are shown to differ by 5 cm of water column. Find the volume of water flowing through the pipe per second.

Solution: