BS111
Probability and Statistics

Lecture 06

Chapter 2: Random Variable

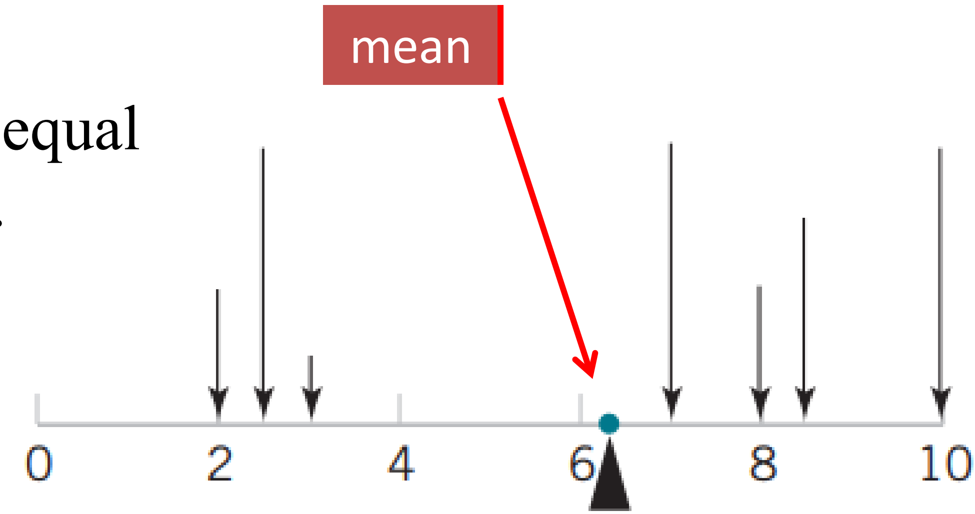
- Discrete Random Variables.
- Probability Mass Functions.
- Cumulative Distribution Functions.
- Discrete R.V. (Mean and Variance.(
- Continuous Random Variables.
- Probability Density Functions.
- Continuous R.V. (Mean and Variance.(
- Joint Probability Distributions.

Mean and Variance ((1/15

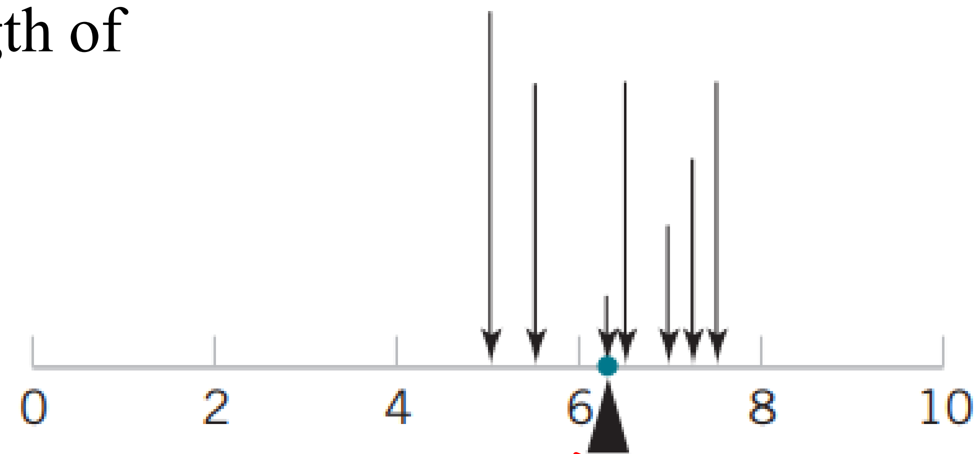
Two numbers are often used to summarize a probability distribution for a random variable X. The mean is a measure of the center or middle of the probability distribution، and the variance is a measure of the dispersion, or variability in the distribution.

Mean and Variance ((2/15

Probability distributions with equal means but different variances.



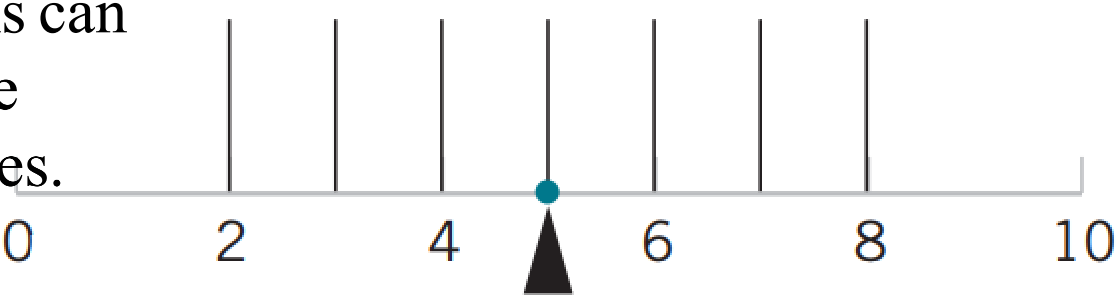
The y-axis represent the probability that shows in length of the arrow

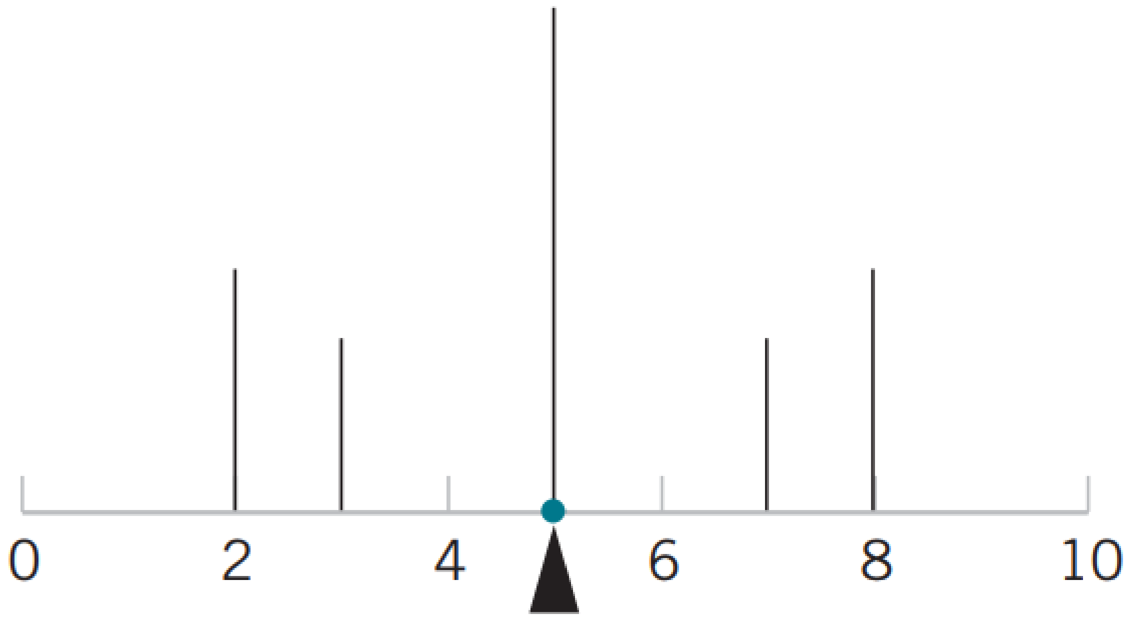


Same mean, but differ distribution

Mean and Variance ((3/15

Two probability distributions can differ even though they have identical means and variances.





Mean and Variance ((4/15

Mean, Variance, and Standard deviation

The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_x x f(x)$$

The **variance** of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_x (x-\mu)^2 f(x) = \sum_x x^2 f(x) - \mu^2$$

$$E(X^2) - (E(X))^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Mean and Variance ((5/15

Example1

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Find:

Determine the mean and variance of the random variable X

Mean and Variance ((6/15

Example1

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Find:

Determine the mean and variance of the random variable X

Answer: ((1/2

$E(X) =$

$$\sum x_i P(x_i) = (-2)\left(\frac{1}{8}\right) + (-1)\left(\frac{2}{8}\right) + (0)\left(\frac{2}{8}\right) + (1)\left(\frac{2}{8}\right) + (2)\left(\frac{1}{8}\right)$$

$$= 0$$

Mean and Variance ((6/15

Example1

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Answer: (2/2)

$$V(X) = E(X^2) - \left(E(X)\right)^2$$

$$E(X) = 0$$

$$E(X^2)$$

$$= \sum x_i^2 P(x_i) = (4)\left(\frac{1}{8}\right) + (1)\left(\frac{2}{8}\right) + (0)\left(\frac{2}{8}\right) + (1)\left(\frac{2}{8}\right) + (4)\left(\frac{1}{8}\right) = 1.5$$

$$V(X) = 1.5 - (0)^2 = 1.5, \qquad \text{Standard Deviation } (\sigma) = \sqrt{1.5}$$

Mean and Variance ((7/15

Example:2

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Mean and Variance ((8/15

Example2 – Answer ((1/9

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Let X represent the number of good components in the sample. **Then x can only take the numbers 0, 1, 2 and .3**

Mean and Variance ((8/15

Example2 – Answer ((2/9

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

Mean and Variance ((8/15

Example2 – Answer ((3/9

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

$$f(0) = P(X = 0) = \frac{\binom{4}{0}\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}$$

Mean and Variance ((8/15

Example2 – Answer ((4/9

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

$$f(1) = P(X = 1) = \frac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}$$

Mean and Variance ((8/15

Example2 – Answer ((5/9

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

$$f(2) = P(X = 2) = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}$$

Mean and Variance ((8/15

Example2 – Answer ((6/9

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

$$f(3) = P(X = 3) = \frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}$$

Mean and Variance ((8/15

Example2 – Answer ((7/9

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

x	0	1	2	3
$f(x) = P(X = x)$	1/35	12/35	18/35	4/35

Mean and Variance ((8/15

Example2 – Answer ((8/9

Find the expected value of the number of good components in this sample.

x	0	1	2	3
$f(x) = P(X = x)$	1/35	12/35	18/35	4/35

$$E(X) = (0)\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right) + (2)\left(\frac{18}{35}\right) + (3)\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7.$$

Mean and Variance ((8/15

Example2 – Answer ((9/9

$E(X) = 1.7$

Determine the variance of the random variable X

x	0	1	2	3
$f(x) = P(X = x)$	1/35	12/35	18/35	4/35

$$V(X) = E(X^2) - \left(E(X)\right)^2$$

$$E(X^2) = \sum x_i^2 P(x_i) = 0\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right) + (4)\left(\frac{18}{35}\right) + (9)\left(\frac{4}{35}\right) = \frac{120}{35} = 3.43$$

$$V(X) = 3.43 - (1.7)^2 = 0.54, \qquad \text{Standard Deviation } (\sigma) = \sqrt{0.54} = 0.74$$

Mean and Variance ((9/15

For any constants *a* and *b*:

Mean

1. $E(a) = a\,, \qquad a \in \mathbb{R}$
2. $E(aX + b) = aE(X) + b\,, \qquad a, b \in \mathbb{R}$

Variance

1. $V(a) = 0\,, \; a \in \mathbb{R}$
2. $V(aX + b) = a^2 V(X)\,, \qquad a, b \in \mathbb{R}$

Mean and Variance ((10/15

Example3: – Answer

A discrete random variable with $V(X) = 2.5$
Evaluate $V(2X + 1)$

$$V(aX + b) = a^2V(X), \qquad a, b \in \mathbb{R}$$

$$V(2X + 1) = 4V(X) = 4 \times 2.5 = 10$$

Mean and Variance ((12/15

Example 4 : – Answer

A discrete random variable with $E(X) = 2.5$
Evaluate $E(2X + 1)$

$$E(aX + b) = aE(X) + b, \qquad a, b \in \mathbb{R}$$

$$E(2X + 1) = 2E(X) + 1$$

$$E(2X + 1) = 2 \times 2.5 + 1 = 6$$

Mean and Variance ((14/15

Example:5 – Answer

Let X is a random variable with mean 6 and variance 100. Consider another random variable Y such that $Y = 3X + 6$, evaluate the mean and variance of Y?

$$E(X) = 6 \quad , \quad V(X) = 100$$

$$E(Y) = E(3X + 6) = 3E(X) + 6 = 3(6) + 6 = 24$$

$$V(Y) = V(3X + 6) = 9V(X) = 9(100) = 900$$

Continuous R. V. ((1/3

Continuous Random Variable:

If the range space R_x of the random variable X is an interval or a collection of intervals, X is called a ***continuous random variable***.

A continuous random variable has a probability of **0** of assuming ***exactly*** any of its values. Consequently, its probability distribution cannot be given in tabular form.

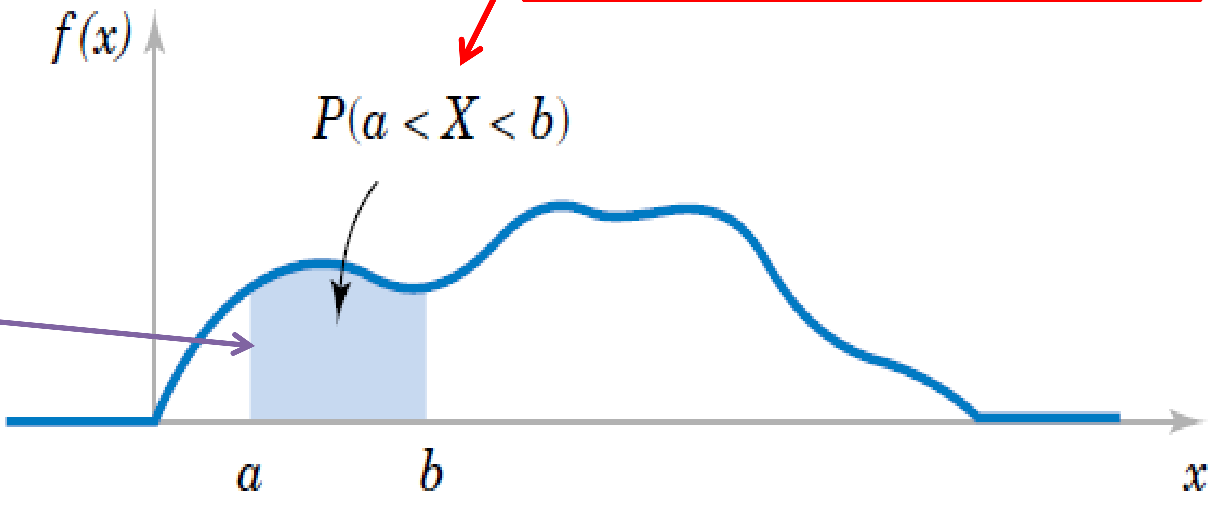
Continuous R. V. ((2/3

Example:

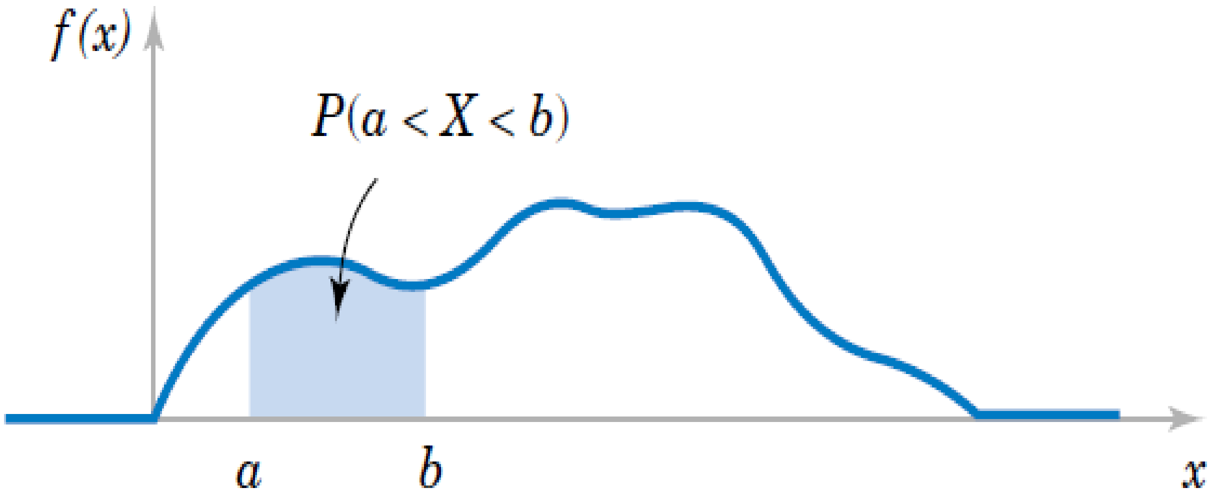
If we talk about the probability of selecting a person who is at least 163 centimeters but not more than 165 centimeters tall. Now we are dealing with an interval rather than a point value of our random variable.

X should be between two values

F(x) is probability and it Represented as AREA



Continuous R. V. ((3/3



If X is a **continuous random variable,** for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2)$$

Prob. Density Functions ((1/6

Probability Density Function

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx =$ area under $f(x)$ from a to b

for any a and b

Prob. Density Functions ((2/6

Definite Integral:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

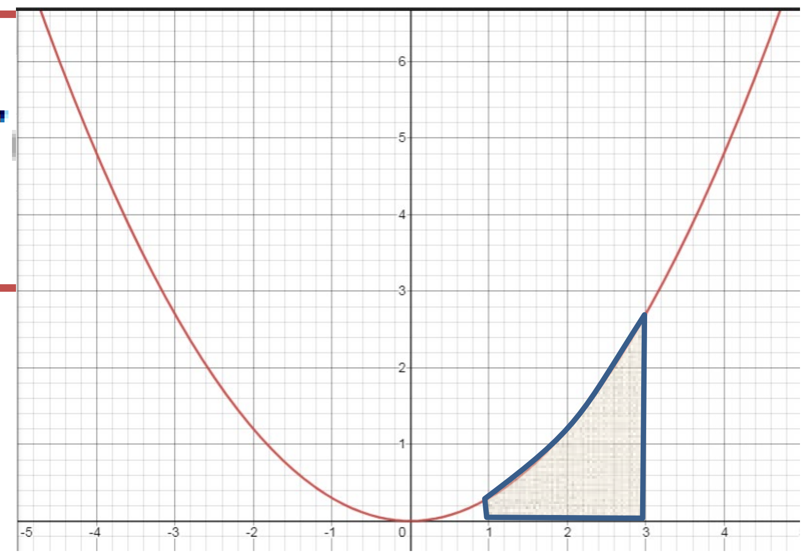
$$\int_1^3 x^2\,dx$$

$$\int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = \left[\frac{3^3}{3}\right] - \left[\frac{1^3}{3}\right] = \left[9 - \left(\frac{1}{3}\right)\right] = \frac{26}{3}$$

Prob. Density Functions ((2/6

Definite Integral:

$$\int_a^b f(x)\,dx = F$$



$$\int_1^3 x^2\,dx$$

$$\int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = \left[\frac{3^3}{3}\right] - \left[\frac{1^3}{3}\right] = \left[9 - \left(\frac{1}{3}\right)\right] = \frac{26}{3}$$

Prob. Density Functions ((3/6

Example:1

Suppose that $\boldsymbol{f(x) = e^{-x}}$ for $\boldsymbol{x} \geq 0$
Check the probability density function, then determine the following probabilities:

1. $P(X < 1)$
2. $P(1 \leq X < 2.5)$
3. $P(X = 3)$
4. $P(X \geq 3)$

Prob. Density Functions ((4/6

Example1 – Answer ((1/5

Check the probability density function:

$$\int_0^{\infty} e^{-x}\, dx$$

When the result = 1
➔ Proved that this function is
probability density function

$$\int_0^{\infty} e^{-x}\, dx = -e^{-x}\Big|_0^{\infty} = \left(-e^{-\infty}\right) - \left(-e^{0}\right) = 0 + 1 = 1$$

Prob. Density Functions ((4/6

Example1 – Answer ((2/5

We should always remember the original interval

(**0 to ∞**)

1) $P(X < 1)$

$$P(X<1)=\int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1 = \left(-e^{-1}\right)-\left(-e^{0}\right)$$

$$= -0.367879 + 1 = 0.632121$$

We should always remember that the result is "**Probability**" and must be **+ve** and between **0** and **1**

Prob. Density Functions ((4/6

Example1 – Answer ((3/5

We should always remember the original interval

(**0 to ∞**)

2) $P(1 \leq X < 2.5)$

$$P(1 \leq X < 2.5) = \int_{1}^{2.5} e^{-x}\,dx = -e^{-x}\Big|_{1}^{2.5}$$

$$= \left(-e^{-2.5}\right) - \left(-e^{-1}\right) = -0.082085 + 0.367879$$

$$= 0.285794$$

Prob. Density Functions ((4/6

Example1 – Answer ((4/5

$3)\ P(X = 3)$

$$P(X = 3) = 0$$

Prob. Density Functions ((4/6

Example1 – Answer ((5/5

4) $P(X \geq 3)$

$$P(X \geq 3) = \int_{3}^{\infty} e^{-x}\, dx = -e^{-x} \Big|_{3}^{\infty} = \left(-e^{-\infty}\right) - \left(-e^{-3}\right)$$

$$= 0 + 0.049787 = 0.049787$$

Prob. Density Functions ((5/6

Example:2

Suppose that the error in the reaction temperature, in °C (Celsius), for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that $f(x)$ is a density function.

(b) Find $P(0 < X \le 1)$.

Prob. Density Functions ((6/6

Example2 – Answer ((1/2

Check the probability density function:

$$\int_{-1}^{2} \frac{x^2}{3}\, dx$$

$$\int_{-1}^{2} \frac{x^2}{3}\, dx = \frac{x^3}{9}\Big|_{-1}^{2} = \left(\frac{(2)^3}{9}\right) - \left(\frac{(-1)^3}{9}\right) = \frac{8}{9} + \frac{1}{9} = \mathbf{1}$$

Prob. Density Functions ((6/6

Example2 – Answer ((2/2

2) $P(0 < X \leq 1)$

$$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3}\, dx = \frac{x^3}{9}\Big|_0^1$$

$$= \left(\frac{(1)^3}{9}\right) - \left(\frac{(0)^3}{9}\right) = \frac{1}{9} + \frac{0}{9} = \frac{1}{9}$$

Joint Prob. Distributions ((1/9

Definition:

If X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a joint probability distribution.

Joint Prob. Distributions ((2/9

If X and Y are discrete random variables:

Joint Probability Mass Function

The **joint probability mass function** of the discrete random variables X and Y, denoted as $f_{XY}(x, y)$, satisfies

(1) $f_{XY}(x, y) \geq 0$

(2) $\sum_{X} \sum_{Y} f_{XY}(x, y) = 1$

(3) $f_{XY}(x, y) = P(X = x, Y = y)$

Joint Prob. Distributions ((3/9

Example:1

A coin is tossed two times. Calculate the joint probability mass function of the discrete random variables X and Y, where X denote the number of heads appear and Y denote the number of tails appear.

Joint Prob. Distributions ((4/9

Example1 – Answer ((1/2

$$S = \{HH, TH, HT, TT\}$$

X	2	1	1	0
Y	0	1	1	2

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Joint Prob. Distributions ((4/9

Example1 – Answer ((2/2

$S = \{HH, TH, HT, TT\}$

X 2 1 1 0

Y 0 1 1 2

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Joint Prob. Distributions ((5/9

Example:2

A coin is tossed two times. Calculate the joint probability mass function of the discrete random variables X and Y, where X denote the number of heads appear and Y denote the number of tails appear.

Find:

1) $f_{XY}(1,2) = P(X = 1,\ Y = 2)$
2) $f_{XY}(2,0) = P(X = 2,\ Y = 0)$
3) $P(X = 1,\ Y \leq 2)$
4) $P(Y = 2)$

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Joint Prob. Distributions ((6/9

Example2 – Answer ((1/4

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

1) $f_{XY}(1,2) = P(X = 1,\ Y = 2) = 0$

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Joint Prob. Distributions ((6/9

Example2 – Answer ((2/4

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

2) $f_{XY}(2,0) = P(X = 2,\ Y = 0)$

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Joint Prob. Distributions ((6/9

Example2 – Answer ((2/4

$y \backslash x$	0	1	2
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

2) $f_{XY}(2,0) = P(X = 2,\ Y = 0) = 1/4$

Joint Prob. Distributions ((6/9

Example2 – Answer ((3/4

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

(3 $P(X = 1, \; Y \leq 2)$

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Joint Prob. Distributions ((6/9

Example2 – Answer ((3/4

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

3) $P(X = 1,\ Y \leq 2) = 0 + \frac{2}{4} + 0 = \frac{2}{4}$

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Joint Prob. Distributions ((6/9

Example2 – Answer ((4/4

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

(4 $P(Y=2)$

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Joint Prob. Distributions ((6/9

Example2 – Answer ((4/4

y \ x	0	1	2
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

4) $P(Y = 2) = \frac{1}{4} + 0 + 0 = \frac{1}{4}$

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Joint Prob. Distributions ((7/9

Marginal Probability Distributions

The marginal distributions of the random variable X alone is:

$$f_X(x) = \sum_y f_{XY}(x, y)$$

The marginal distributions of the random variable Y alone is:

$$f_Y(y) = \sum_x f_{XY}(x, y)$$

Joint Prob. Distributions ((8/9

Example:3

A coin is tossed two times. Calculate the joint probability mass function of the discrete random variables X and Y, where X denote the number of heads appear and Y denote the number of tails appear.

Find:

1) $f_X(x)$

2) $f_Y(y)$

Joint Prob. Distributions ((9/9

Example3 – Answer ((1/4

y \ *x*	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

1) $f_X(x)$

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Joint Prob. Distributions ((9/9

Example3 – Answer ((1/4

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0
$f_X(x)$	**4/1**	**4/2**	**4/1**

Find:

1) $f_X(x)$

Joint Prob. Distributions ((9/9

Example3 – Answer ((2/4

x	0	1	2
$f_X(x)$	4/1	4/2	4/1

Find:

1) $f_X(x)$

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Joint Prob. Distributions ((9/9

Example3 – Answer ((3/4

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

2) $f_Y(y)$

Joint Prob. Distributions ((9/9

Example3 – Answer ((3/4

y \ x	**0**	**1**	**2**	$f_Y(y)$
0	0	0	1/4	**4/1**
1	0	2/4	0	**4/2**
2	1/4	0	0	**4/1**

Find:

2) $f_Y(y)$

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Joint Prob. Distributions ((9/9

Example3 – Answer ((4/4

y	0	1	2
$f_Y(y)$	4/1	4/2	4/1

Find:

2) $f_Y(y)$

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Thank You