Linear Algebra (II)

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Instructor

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Chapter 1: Linear transformations

1.1 Examples and elementary properties.

Definition 1.1.1:

Let V and W be two vector spaces over the same field F. A mapping $T:V\to W$ is called a linear transformation if it is satisfied the following axioms

1. $T(v_1+v_2)=T(v_1)+_W T(v_2)$, where $v_1,v_2\in V$.
2. $T(\lambda v)=\lambda T(v)$, where $v\in V$ and $\lambda\in F$.

In a linear transformation T, if $V=W$, then T is called a linear operator on V.

Theorem 1.1.2:

Let V and W be two vector spaces over the same field F. A mapping $T:V\to W$ is linear transformation if and only if

$$v_1,v_2\in V \wedge \lambda,\mu\in F\to T(\lambda v_1+\mu v_2)=\lambda T(v_1)+\mu T(v_2).$$

Example 1.1.3:

i. Consider the vector spaces $\Re^2$ and $\Re^3$ over $\Re$, define

1. $T: \Re^2\to\Re^3$ such that $T(x, y)=(x+y, 3x-4y, 2y)$
2. $T: \Re^2\to\Re^3$ such that $T(x, y)=(x-y, x+y-1, x)$
3. $T: \Re^2\to\Re^2$ such that $T(x, y)=(xy, x)$

ii. Consider the vector spaces P_{n-1}, P_n and P_{n+1}

1. $T:P_n\to P_{n-1}$ such that $T(p(x))=\frac{dp(x)}{dx}$
2. $T:P_n\to P_{n+1}$ such that

$$T\big(p(x)\big)=\int_0^x p(t)dt$$

Theorem 1.1.4:

Let $T:V\to W$ be a linear transformation, then

1. $T(0)=0$.
2. $u\in V\to T(-u)=-T(u)$,

Definition 1.1.5:

1. The linear operator $I_V:V\to V$ such that $I_V(v)=v$, $\forall v\in V$ is called the identity operator on V.
2. The linear transformation $0:V\to W$ such that $0(v)=0$, $\forall v\in V$ is called the zero transformation.

Theorem 1.1.6:

Let $T,S:V\to W$ be two linear transformations such that $V=\text{span}(\{v_1,\ldots, v_n\})$. If $T(v_i)=S(v_i)$, $\forall i$, $1\leq i\leq n$, then $T=S$.

Exercise 1.1.7:

i. Verify whether the following functions are linear transformations or not.
   1. $T:\Re^2\to\Re^2$ such that $T(x, y)=(x,-y)$.
   2. $T:C\to C$ such that $T(z)=\bar{z}$.
   3. $S:M_{n\times n}\to M_{n\times n}$ such that $S(A)=A+A^T$.

ii. Let $T:V\to W$ be a linear transformation, $\{v_1,\ldots,v_n\}\subseteq V$. If $\{T(v_1),\ldots,T(v_n)\}$ is linearly independent, then so is $\{v_1,\ldots,v_n\}$.

1.2Kernel and image of linear transformation

Definition 1.2.1:

Let $T:V\to W$ be a linear transformation, the kernel and the image of T is defined as follows

$$ker(T)=\{v\in V|T(v)=0\}.$$

$$img(T)=\{w\in W|\ w=T(v), \text{ for some } v\in V\}.$$

Example 1.2.2:

Let $T:\Re^3 \rightarrow \Re^3$ be a mapping defined by $T(x, y, z)=(x-y, z, y-x)$.

1. Show that T is a linear transformation.
2. Find $ker(T)$ and $img(T)$.

Solution:

$Ker\ (T)=\{(x,x,0)|x\in\Re\}$ and $img\ (T)=\{(a,b,-a)|a,b\in\Re\}$.

Example 1.2.3:

Let $S:M_{n\times n}\rightarrow M_{n\times n}$ be a linear transformation such that $S(A)=A-A^T$. Find $ker(T)$ and $img(T)$.

Theorem 1.2.4:

For any linear transformation $T:V\rightarrow W$, $ker(T)$ is a subspace of V and $img(T)$ is a subspaces of W.

Theorem 1.2.5:

Any linear transformation $T:V\rightarrow W$ is 1-1 if and only if $ker(T)=\{0\}$.

Example 1.2.6:

Let $S:\Re^2\rightarrow\Re^3$ and $T:\Re^3\rightarrow\Re^2$ be linear transformation such that. Apply **Theorem 1.2.5** for each of the following

1. $T\ (x,y,z)=(x+y,x-y)$.
2. $S\ (x,y)=(x+y,x-y,x)$.

Exercise 1.2.7:

Let $T:V\rightarrow W$ be a linear transformation, prove or disprove

1. $Ker(T)=V\rightarrow W=\{0\}$.
2. $V=W\rightarrow Ker(T)\subseteq img(T)$.
3. $W=\{0\}\subseteq Ker(T)=V$.
4. $V=W \wedge img(T)\subseteq ker(T)\rightarrow T=0$.

1.3 Isomorphism and composition

Definition 1.3.1:

A linear transformation $T:V\rightarrow W$ is called isomorphism if T is 1-1 and onto. Two vector spaces are called isomorphic if there is an isomorphism between them. We use the notation $V\cong W$.

Example 1.3.2:

1. The identity linear transformation on V is isomorphism.
2. $S:M_{n\times n}\rightarrow M_{n\times n}$ such that $S(A)=A^T$ is an isomorphism.
3. The zero linear transformation $T:V\rightarrow W$ is not isomorphism.

Theorem 1.3.3:

Let $T:V\rightarrow W$ be a linear transformation for which V and W are finite dimensional. Then the following statements are equivalent

1. T is an isomorphism.
2. If $\{v_1,\ldots,v_n\}$ is a basis of V, then $\{T(v_1),\ldots, T(v_n)\}$ is a basis of W.
3. There exists a base $\{v_1,\ldots,v_n\}$ of V, such that $\{T(v_1),\ldots, T(v_n)\}$ is a basis of W.

Theorem 1.3.4:

Let V and W be two finite dimensional vector spaces. Then

$$V\cong W \leftrightarrow dim(V)=dim(W).$$

Proof:

Let $V\cong W$, then $\exists$ an isomorphism $T:V\rightarrow W$.

Suppose $\{r_1,\ldots,r_m\}$ be a base of V, then by **Theorem 1.3.3**, $\{T(r_1),\ldots, T(r_m)\}$ is a basis of W.

On the other hand, $T(r_i)\neq T(r_j)$, since T is 1-1, then $dim(V)=dim(W)$.

Conversely, let $\{v_1,\ldots,v_n\}$ and $\{w_1,\ldots,w_n\}$ be basis for V and W respectively.

Define $T:V\rightarrow W$ as follows

1. $T(v_i)=w_i$.

2. For $v\in V$, we have $v=a_1v_1+\ldots+a_nv_n$ then we define

   $T(v)= a_1T(v_1)+\ldots+a_nT(v_n)=a_1w_1+\ldots+a_nw_n$.

Is T well defined?

$v_i=v_j\rightarrow T(v_i)=T(v_j)$.

Otherwise $dim(V)\neq dim(W)$.

$v=v^*\rightarrow v= a_1v_1+\ldots+a_nv_n=v^*\rightarrow T(v)= a_1T(v_1)+\ldots+a_nT(v_n)=T(v^*)$.

To show that T is a linear transformation,

1. Let $v, v^*\in V$, then

$v=b_1v_1+\ldots+b_nv_n$ and $v^*=c_1v_1+\ldots+c_nv_n$.

$T(v+v^*)=T\big((b_1v_1+\ldots+b_nv_n)+(c_1v_1+\ldots+c_nv_n)\big)$

$=T\big((b_1+c_1)v_1+\ldots+(b_n+c_n)v_n\big)$

$=(b_1+c_1)T(v_1)+\ldots+(b_n+c_n)T(v_n)$

$=\big(b_1T(v_1)+\ldots+b_nT(v_n)\big)+\big(c_1T(v_1)+\ldots+c_nT(v_n)\big)$

$=T(v)+T(v^*)$

2. Let $v\in V$ and λ be a scalar,

   $T(\lambda v)=T\big(\lambda(a_1v_1+\ldots+a_nv_n)\big)=T(\lambda a_1v_1+\ldots+\lambda a_nv_n)= \lambda a_1T(v_1)+\ldots+\lambda a_nT(v_n)$

   $=\lambda(a_1T(v_1)+\ldots+a_nT(v_n))=\lambda T(v)$.

For 1-1,

Let $T(v)=T(v^*)$

$T(v)=T(v^*)\rightarrow v=v^*$ (Homework).

For onto,

Let $w\in W$

$w\in W\rightarrow w=a_1w_1+\ldots+a_nw_n=a_1T(v_1)+\ldots+a_nT(v_n)=T(a_1v_1+\ldots+a_nv_n)= T(v)$, for some $v\in V$.

Example 1.3.5:

P_2 and the set of all symmetric matrices of order two are isomorphic.

Dimension Theorem:

Let $T:V\to W$ be al linear transformation for which $ker(T)$ and $img(T)$ are finite dimensional, then V is a finite dimensional and

$$dim\ (ker\ (T))+dim(img(T))=dim\ (V)$$

Proof:

Not required.

Theorem 1.3.6:

Let V and W be two finite dimensional vector spaces of the same order. A linear transformation $T:V\to W$ is isomorphism if T is either 1-1 or onto.

Exercise 1.3.7:

Let $T:\Re^3\to\Re^3$ be a linear transformation such that $T(x,\ y,\ z)=(x+y,y+z,x+z)$. Is T isomorphism?

Theorem 1.3.8:

Let $T:V\to W$ and $S:W\to Z$ be linear transformations, then

1. $S\circ T$ is a linear transformation, where $S\circ T$ is the composition function of T and S.
2. If T and S are isomorphism, then so is $S\circ T$.

Proof:

$S\circ T\ (v_1+v_2)= S(T\ (v_1+v_2))$ (Definition of composition of functions)

$= S(T\ (v_1)+T\ (v_2))$ (T is L.T.)

$= S(T\ (v_1))+S\ (T\ (v_2))$ (S is L.T.)

$= S\circ T\ (v_1)+S\circ T\ (v_2)$ (Why?)

$S\circ T\ (\lambda v)=S(T\ (\lambda v))= S(\lambda T\ (v))= \lambda S(T\ (v))=\lambda S\circ T\ (v)$

2. S and T are bijective$\to$ $S\circ T$ is bijective.

Then form 1. and 2. $S\circ T$ is isomorphism.

Definition 1.3.9:

Let V and W be vector spaces, $T:V\to W$ and $S:W\to V$ be linear transformations. We say that S is the inverse of T if

$$T\circ S=I_W \text{ and } S\circ T=I_V.$$

The inverse of T is denoted by T^{-1}.

A linear transformation T is called invertible if it has the inverse.

Notice that if S is the inverse of T, then T is the inverse of S.

Exercise 1.3.10:

A linear transformation T has the inverse if and only if T is 1-1 and onto.

Example 1.3.11:

Verify whether the following transformation is invertible or not.

1. $T:\Re^3\to\Re^3$ such that $T(x, y, z)=(x+y,y+z, x+z)$

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & u-v+w \\ 0 & -2 & 0 & -u-v+w \\ 0 & 0 & 2 & -u+v+w \end{array}\right)$$

2. $T:\Re^4\to\Re^4$ such that $T(x, y, z,w)=(x+y,y+z, z+w,x+w)$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.3.12:

i. Find a linear transformation with the given properties and compute $T(v)$:

   1. $T:\Re^2\to\Re^3$; $T(1,2)=(1,0,1)$, $T(-1,0)=(0,1,1)$, $v=(-3,2)$
   2. $T:P_2\to P_3$, $T(x^2)=x^3$, $T(x+1)=0$, $T(x-1)=x$, $v=x^2+x+1$.

Remark:

Step 1: Show that {(1,2),(-1,0)} is a base of $\Re^2$.

Step 2: Write v=(-3,2) as a linear combination of (1,2) and (-1,0).

(-3,2)=1(1,2)+4(-1,0)

Step 3: Take T for Step 2,

T (-3,2)=$T\big($1(1,2)+4(-1,0)$\big)$

=1T (1,2)+4T(-1,0)

=1(1,0,1)+4(0,1,1)=(1,4,5).

ii. Verify whether the following linear transformations are isomorphism or not

1. $T:P_1\to\Re^2$ such that $T(p(x))=(p(0), p(1))$.
2. $T:V\to V$, $T(v)=\lambda v$, λ is a nonzero scalar.
3. $T:P_2\to P_2$ such that $T(p(x))=p(x+1)$.

iii. Is the linear transformation T that is defined in **Exercise 1.3.12**, *i*. isomorphism?

1.4 Operations with linear transformations

Definition 1.4.1:

Let $T:V\to W$ and $S:V\to W$ be linear transformations. Define

1. The sum ($T\oplus S$) of T and S as a function from V to W as follows

$$(T\oplus S)(v)= T(v)+S(v).\ \forall\ v_1,v_2\in V.$$

2. The scalar product λT from V to W as follows:

$$(\lambda\odot T)(v)=\lambda T(v).\ \forall\ v\in V \text{ and } \lambda\in F.$$

Theorem 1.4.2:

Let V and W be vector spaces over the same field F, then the collection of all linear transformations with the operations defined in **Definition 1.4.1** is a vector space denoted by Hom(V,W).

Proof:

Hom(V,W)={T|T:$V\rightarrow W$ is a linear transformation}

We have to show that Hom(V,W) with the operations $\oplus$ and $\odot$ is a vector space over the filed F.

For associativity $(T\oplus S)\oplus U=T\oplus(S\oplus U)$

$(T\oplus S)\oplus U(v)= (T\oplus S)(v)\oplus U(v)= \big(T(v)\oplus S(v)\big) \oplus U(v)= T(v)\oplus\big(S(v) \oplus U(v)\big)= T(v)\oplus\big(S \oplus U(v)\big)$

$=T\oplus(S\oplus U)(v)$.

For commutativity, homework

The zero transformation is the identity.

For any T:$V\rightarrow W$, $-T$:$V\rightarrow W$ is the inverse of T.

For $(\lambda+\mu)\odot T=(\lambda+\mu)\odot T$?

$(\lambda+\mu)\odot T(v)= (\lambda+\mu)T(v)= \lambda T(v)+\mu T(v)= \lambda\odot T(v)+\mu \odot T(v)$.

The others are homework.

Theorem 1.4.3:

Let V and W be two vector space such that $dim(V)=m$ and $dim(W)=n$. Then $dim(\text{Hom}(V,W))=mn$.

Proof: Not required.

1.5 Matrix representation of a linear transformation

Definition 1.5.1:

Let $A_{m\times n}$ be a matrix. The matrix transformation $T_A:\Re^n\rightarrow\Re^m$ is defined by

$$T_A(X)=AX, \text{ where } X\in\Re^n.$$

Theorem 1.5.2:

For each matrix $A_{m\times n}$, the matrix transformation T_A defined in **Definition 1.5.1** is a linear transformation.

Proof:

$T_A(\lambda X+\mu Y)=A(\lambda X+\mu Y)$ (By **Definition 1.5.1**)

$=A(\lambda X)+A(\mu Y)$ (Matrix property)

$=\lambda AX+\mu AY$ (Matrix property)

$=\lambda T_A(X)+\mu T_A(Y)$

Example 1.5.3:

For each of the following matrices, find T_A

1. $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
2. $A=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

Solution:

1. $T_A(X)=T_A(\begin{pmatrix}x\\ y\end{pmatrix})=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}y\\ x\end{pmatrix}$
2. $T_A(X)=T_A(\begin{pmatrix}x\\ y\end{pmatrix})=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}2x\\ y\end{pmatrix}$

Exercise 1.5.4:

Find the matrix transformation of the matrix $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$.

Chapter 2: Eigenvalues and Digonalisation

2.1 Eigenvalues and similarity

Definition 2.1.1:

Let A be a square matrix of order n. A scalar λ (real or complex) is said to be an eigenvalue of A if,

$$\exists \text{ a nonzero column vector } X \text{ such that } AX=\lambda X. \quad \ldots(2.1)$$

In this case, X is called an eigenvector of A corresponding to λ.

Definition 2.1.2:

Let λ be an eigenvalue of the matrix $A_{n\times n}$. The set of all eigenvectors defined in **Definition 2.1.1** is called the eigenspace associated to λ, i.e.

$$E_\lambda(A)=\{\, X \mid AX=\lambda X\}.$$

Theorem 2.1.3:

For each λ, the set $E_\lambda(A)$ is a subspace of ($\Re^n$ or C^n).

Remark 2.1.4:

The equation (2.1) is the same as the equation $(A-\lambda I_n)X=0$.

Definition 2.1.5:

The determinant of the equation $A-\lambda I_n=0$ is called the characteristic polynomial of the matrix $A_{n\times n}$, and denoted by $c_A(\lambda)$,

$$c_A(\lambda)=det(A-\lambda I_n)$$

Clearly, the eigenvalues of a matrix A is the roots (zeros) of the characteristic polynomial and vice versa.

Example 2.1.6:

Find the eigenvalues and the eigenspace of the following matrices

1. $A=\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}$
2. $A=\begin{pmatrix}1 & -2 & 3\\ 2 & 6 & -6\\ 1 & 2 & -1\end{pmatrix}$

Solution:

$p(\lambda)=|A-\lambda I_2|=\left|\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=\begin{vmatrix}2-\lambda & 3\\ 1 & 4-\lambda\end{vmatrix}=\lambda^2-6\lambda+5.$

The eigenvalues are $\lambda=5$ and $\lambda=1$.

For the eigenvector of $\lambda=5$, we apply

$AX=\lambda X$, then

$$\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}\longrightarrow\begin{pmatrix}2x+3y\\ x+4y\end{pmatrix}=\begin{pmatrix}5x\\ 5y\end{pmatrix}$$

Then, we obtain $y=x$.

So, the eigenvector of $\lambda=5$ is $\begin{pmatrix}x\\ x\end{pmatrix}$

$E_5(\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix})=\text{span}\left\{\begin{pmatrix}1\\ 1\end{pmatrix}\right\}$.

For the eigenvector of $\lambda=1$,

$$\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=1\begin{pmatrix}x\\ y\end{pmatrix}\longrightarrow\begin{pmatrix}2x+3y\\ x+4y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

Then, we obtain $y=\frac{-x}{3}$.

So, the eigenvector of $\lambda=5$ is $\begin{pmatrix}-3\\ 1\end{pmatrix}$

$E_1(\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix})=\text{span}\left\{\begin{pmatrix}-3\\ 2\end{pmatrix}\right\}$.

Definition 2.1.7:

An eigenvalue λ is said to be of multiplicity m if it is repeated m times.

$$c_A(\lambda)=(x-\lambda)^m q(x)$$

Example 2.1.8:

$$A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & 11 \end{pmatrix}$$

$\lambda=1$ and $\lambda=-3$ (with multiplicity two) are the eigenvalues of A.

$E_1(A)=\text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}$ and $E_{-3}(A)=\text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Exercise 2.1.9:

Find the eigenvalues and the eigenspace of the following matrix.

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{pmatrix}$$

Theorem 2.1.10:

If λ is an eigenvalue of a matrix A with the nonzero eigenvector X, then λ^2 is an eigenvalue of the matrix A^2 with the same eigenvector X.

Proof:

$|A^2-\lambda^2 I|=|A^2-\lambda^2 I^2|=|(A-\lambda I)(A+\lambda I)|= |A-\lambda I|\ |A+\lambda I|=0.\ |A+\lambda I|=0.$

Then λ^2 is an eigenvalue of A^2.

$A^2X= A(AX)=A(\lambda X)=\lambda(AX)=\lambda(\lambda X)=\lambda^2 X.$

Then X is the eigenvector of A^2 corresponding to λ^2.

Exercise 2.1.11:

Regarding to **Theorem 2.1.10,** show that $\lambda^3-2\lambda+3$ is an eigenvalue of the matrix $A^3-2A+3I$.

Example 2.1.12:

For a triangular matrix $A=(a_{ij})$, the set of eigenvalues are the entries of the main diagonal.

Solution:

Let A be an upper triangular matrix of order n. Then,

$$c_A(\lambda) = |A - \lambda I_n| = \prod_{i=1}^{n}(a_{ii} - \lambda)$$

Then $\lambda = a_{ii}$, for all $i=1,\dots,n$.

Similarly, for the lower triangular matrix.

Example 2.1.13:

Prove that A and A^T have the same eigenvalues.

Solution:

Let A be a square matrix of order n. Then,

for A^T, the characteristic polynomial is given by,

$$c_{A^T}(\lambda) = |A^T - \lambda I_n| = |(A - \lambda I_n)^T| = |A - \lambda I_n|.$$

Definition 2.1.14:

Let A and B be two square matrices of the same order, we say that A and B are similar if $B = P^{-1}AP$ or $B = PAP^{-1}$, for some invertible matrix P.

We use the expression ($A \sim B$) for two similar matrices A and B.

Example 2.1.15:

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$. Show that $A \sim B$ if $B = \begin{pmatrix} -2 & 5 \\ -1 & 3 \end{pmatrix}$.

Solution:

We may select $P = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$. Then

$$P^{-1}AP = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ -1 & 3 \end{pmatrix}$$

Theorem 2.1.16:

Let $A \sim B$, then

1. $A^{-1} \sim B^{-1}$.

2. $\lambda A \sim \lambda B$.
3. $A^T \sim B^T$.

Proof:

1. $A \sim B \to \exists$ an invertible matrix P such that $B=P^{-1}AP$.

$\to \exists$ an invertible matrix P such that $B^{-1}=(P^{-1}AP)^{-1}$

$\to \exists$ an invertible matrix P such that $B^{-1}=P^{-1}A^{-1}(P^{-1})^{-1}$

$\to \exists$ an invertible matrix P such that $B^{-1}=P^{-1}A^{-1}P$.

Theorem 2.1.17:

Let A and B be two similar matrices, then

1. A and B have the same determinant.
2. A and B have the same trace.
3. A and B have the same characteristics polynomial.
4. A and B have the same eigenvalues.

Proof:

1. $|B|=|P^{-1}||A||P|$.
2. $trace(B)= trace(P^{-1}AP) = trace(AP^{-1}P) = trace(AI_n) = trace(A)$.
3. $c_B(\lambda)=|B-\lambda I|=|P^{-1}AP-\lambda I|=|P^{-1}AP-\lambda P^{-1}P|=|P^{-1}AP-P^{-1}\lambda P|=|P^{-1}(AP-\lambda P)|=|P^{-1}|\,|AP-\lambda P|$

$=|P^{-1}|\,|(A-\lambda I)P|=|P^{-1}|\,|A-\lambda I|\,|P|==|A-\lambda I|=c_A(\lambda)$.

Theorem 2.1.18:

If $A \sim B$, then $rank\,(A)=rank\,(B)$.

Remark 2.1.19:

1. The converse of all tasks that mentioned in **Theorem 2.1.17** need not be true.
2. The converse of **Theorem 2.1.18** need not be true.

Example 2.1.20:

The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and I_2 are not similar, while,

1. $|A|=|I_2|$.
2. $trace(A)= trace(I_2)$.
3. $rank(A)= rank(I_2)$.
4. The eigenvalue of A is $\lambda=1$ with multiplicity 2.

Theorem 2.1.21:

Let Ψ be the set of all square matrices of order n. Define a relation R as follows,

$$R=\{(A,B)\in\Psi\times\Psi|A\sim B\}.$$

Then R is an equivalence relation on Ψ.

Exercise 2.1.22:

i. Find the eigenvalues and eigenspaces of the following matrices,
   1. $A=\begin{pmatrix}1 & 2\\ 3 & 2\end{pmatrix}$
   2. $A=\begin{pmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & -1 & 2\end{pmatrix}$
   3. $A=\begin{pmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{pmatrix}$

ii. Let $A=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. Is $A\sim B$?

2.2 Diagonalisation

Definition 2.2 1:

A real square matrix A of order n is said to be diagonalisable if it is similar to a diagonal matrix. That is,

$$P^{-1}AP \text{ is diagonalisable, for some invertible matrix } P.$$

Theorem 2.2.2:

A square matrix A of order n is diagonalisable if and only if it has n linearly independent eigenvectors.

Proof:

Step 1: Let A be a diagonalisable matrix of order n,

Then

$$D = P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Step 2: We find P by finding each of its columns, $P=(X_1\ X_2\ \dots X_n)$.

Step 3: From Step 1, we have $AP=PD$, then,

$$A(X_1\ X_2\ \dots X_n)= (X_1\ X_2\ \dots X_n) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

$$\to (AX_1\ AX_2\ \dots\ AX_n)=(\lambda_1 X_1\ \lambda_2 X_2\ \dots\ \lambda_n X_n)$$

$$\to AX_i=\lambda_i X_i,\ i=1,\dots,n.$$

So, X_i, $i=1,\dots,n$ are the eigenvectors of A corresponds to the eigenvalues λ_I, $i=1,\dots,n$ respectively.

Previous result: A matrix P is invertible if and only if its columns are linearly independent.

Then X_i, $i=1,\dots,n$ are linearly independent.

Conversely, if X_i, $i=1,\dots,n$ are linearly independent, then $P=(X_1\ X_2\ \dots X_n)$ is invertible, hence we obtain $AP=PD$, or equivalently, $D=P^{-1}AP$.

Diagonalisation Algorithm

Let A be a square matrix of order n,

1. Find the eigenvalues of A.
2. Find n eigenvectors if possible $X_1, \dots, X_n$.
3. Select $P=[\ X_1\ X_2\ \dots\ X_n]$
4. $P^{-1}AP$ is diagonal.

Example 2.2.3:

Determine whether the following matrices are diagonalisable or not

1. $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$
2. $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & 11 \end{pmatrix}$

Solution:

1. For $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

   $\lambda=0$ and $\lambda=4$ are the eigenvalues of A.

   $E_0(A)=\text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$ and $E_4(A)=\text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$.

   $X_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $X_2=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are linearly independent.

   Let $P=\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$.

   Then $P^{-1}=\frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$

   Hence, $D=P^{-1}AP=\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$

2. For $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & 11 \end{pmatrix}$

   $\lambda=1$ and $\lambda=-3$ (with multiplicity two) are the eigenvalues of A.

   $E_1(A)=\text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}$ and $E_{-3}(A)=\text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

   $X_1=\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $X_2=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $X_3=\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent.

   Let $P=\begin{pmatrix} 2 & -1 & -2 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

   Then $P^{-1}=\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -2 \\ 1 & 1 & 3 \end{pmatrix}$

   Hence,

   $D=P^{-1}AP=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

Example 2.2.4:

Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$$

λ=3 and λ=-1 (with multiplicity two) are the eigenvalues of A.

$E_3(A)$=span$\left\{\begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix}\right\}$ and $E_{-1}(A)$=span$\left\{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\}$.

Any other eigenvector X_3 of λ=-1 is linearly dependent with respect to X_2. Then we cannot obtain an invertible matrix P. Hence A is not diagonalisable.

Theorem 2.2.5:

Let $\lambda_1,\ldots,\lambda_n$ be distinct eigenvalues of a square matrix. If $X_1, \ldots, X_n$ are the corresponding eigenvectors to $\lambda_1,\ldots,\lambda_n$, then $\{X_1, \ldots, X_n\}$ is a linearly independent set.

Proof:

We apply mathematical induction on n,

Step 1: n=1, clearly $\{X_1\}$ is linearly independent, since it is a nonzero vector.
Step 2: Suppose it is true for all numbers less than n.
Step 3: For n, let

$$\sum_{i=1}^{n} b_i X_i = 0 \ldots (1),$$

Then,

$$\sum_{i=1}^{n} b_i A X_i = 0,$$

Since $AX_i=\lambda X_i$, for all i=1,..,n., then

$$\sum_{i=1}^{n} b_i \lambda_i X_i = 0 \ldots (2),$$

Let's multiply (1) by λ_1, then, we obtain

$$\sum_{i=1}^{n} b_i \lambda_1 X_i = 0 \ldots (3),$$

Take (2)-(3), we obtain,

$$\sum_{i=2}^{n} b_i(\lambda_i - \lambda_1)X_i = 0 \ldots (4),$$

From Step 2, we obtain,

$b_i(\lambda_i - \lambda_1)=0$, for all $i=1,\ldots,n$.

Since λ_i, are distinct, for all $i=1,\ldots,n$.

Then $b_i=0$, for all $i=1,\ldots,n$.

Theorem 2.2.6:

A square matrix of order *n* with *n* distinct eigenvalues is diagonalisable.

Remark 2.2.7:

The converse of **Theorem 2.2.6** need not be true.

Theorem 2.2.8:

Let *A* be a square matrix of order *n* and

$$c_A(\lambda) = \prod_{i=1}^{n} (x - \lambda_i)^{m_i}$$

be the characteristic polynomial.

If $d_i=dim(E_{\lambda_i}(A))$.

Then the following statements are equivalent:

1. *A* is diagonalisable
2. $$\sum_{i=1}^{n} d_i = n, \forall i = 1, \ldots, n.$$
3. $d_i=m_i$, $\forall i$

Example 2.2.9:

Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 7 & 3 & 3 & 0 \\ 2 & 6 & 4 & 1 \end{pmatrix}.$$

Prove that *A* is not diagonalisable.

Exercise 2.2.10:

i. Show that

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is a diagonalisable matrix

ii. Show that

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is not a diagonalisable matrix.

iii. Prove or disprove:

1. The sum of two diagonalisable matrices is diagonalisable.
2. If *A* is diagonalisable, then so is λA, $\lambda \neq 0$.

Chapter 3: Inner product spaces, Orthogonality

3.1 Inner product spaces (Definition and examples)

Definition 3.1.1:

Let V be a vector spaces over a field $\Re$. An inner product on V is a function that assigns a number $\langle u, v\rangle$ to every pairs $u,v\in V$ such that the following axioms are satisfied:

3. $\langle u, v\rangle=\langle v, u\rangle$. (symmetric property)
4. $\langle au+bw, v\rangle=a\langle u, v\rangle+b\langle w, v\rangle$, $\forall a, b\in(\Re$ or $C)$. (linear property)
5. $\langle u,u\rangle>0$, $\forall u\neq 0$. (positive definite property)

A vector space V with an inner product $\langle\ ,\ \rangle$ is called an inner product space.

Clearly,

$\langle au-bw, v\rangle=a\langle u, v\rangle-b\langle w, v\rangle$.

Example 3.1.2:

1. Consider the vector space $\Re$ over $\Re$. Define $\langle\ ,\ \rangle$ on $\Re$ as follows

$$\langle x, y\rangle=x.y \text{ (The dot product)}$$

2. Consider the vector space $\Re^2$ over $\Re$. Define $\langle\ ,\ \rangle$ on $\Re$ as follows

$$\langle(x_1,y_1), (x_2,y_2)\rangle=x_1.x_2+ y_1.y_2 \text{ (The dot product)}$$

Example 3.1.3:

Consider the vector space $\Re$ over $\Re$. Define $\langle\ ,\ \rangle$ as follows

$$\langle x, y\rangle=|x-y|.$$

$\langle x, y\rangle$ is not an inner product on $\Re$.

Example 3.1.4:

Let $a, b\in\Re$, define

$C[a, b]=\{ f \mid f$ is continuous on the closed interval $[a, b]\}$.

Define $\langle\ ,\ \rangle$ on C[a, b] as follows

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx$$

Then $\langle f, g\rangle$ is an inner product on C[a, b].

Example 3.1.5:

In $\Re^3$ and for every u and $v\in\Re^3$, If

$$\langle u,v\rangle=u^TAv, \text{ where}$$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Then $\langle u,v\rangle$ is an inner product.

Exercise 3.1.6:

5. According to **Example 3.1.2**, **Task 2.,** how can we define an inner product on the vector space $\Re^n$ over $\Re$? Explain your answer.
6. Determine whether the following $\langle\ ,\ \rangle$ on the corresponding vector space is an inner product or not.
   i. V=P_3 with $\langle p(x), q(x)\rangle=p(1)q(1)$
   ii. V=C (the set of complex numbers) with $\langle z, w\rangle=z\overline{w}$
   iii. V=$M_{2\times 2}$ with $\langle A,B\rangle=|AB|$

Theorem 3.1.7:

Let $\langle\ ,\ \rangle$ be an inner product on a vector space V. For any $u,v,w\in V$ and a, $b\in\Re$.

1. $\langle u,av+bw\rangle=a\langle u,v\rangle+b\langle u,w\rangle$
2. $\langle au,v\rangle=a\langle u,v\rangle=\langle u,av\rangle$
3. $\langle u,0\rangle=0=\langle 0,u\rangle$
4. $\langle u,u\rangle=0\leftrightarrow u=0$

Proof:

1. $\langle u,av+bw\rangle=\langle av+bw,u\rangle=a\langle v,u\rangle+b\langle w,u\rangle= a\langle u,v\rangle+b\langle u,w\rangle$.

2. $\langle au,v\rangle=\langle au+0w,v\rangle=a\langle u,v\rangle+0\langle w,v\rangle=a\langle u,v\rangle$.

   Similarly and by Task 1., we obtain $a\langle u,v\rangle=\langle u,av\rangle$

3. $\langle u,0\rangle=\langle u,0+0\rangle=\langle u,1(0)+1(0)\rangle=1\langle u,0\rangle+1\langle u,0\rangle=\langle u,0\rangle+\langle u,0\rangle$

   $\langle u,0\rangle=\langle u,0\rangle+\langle u,0\rangle\rightarrow\langle u,0\rangle-\langle u,0\rangle=(\langle u,0\rangle+\langle u,0\rangle)-\langle u,0\rangle\rightarrow 0=\langle u,0\rangle+0$

4. $\langle u,u\rangle=0\rightarrow u=0$ by **Definition 3.1.1**.

   For $u=0\rightarrow\langle u,u\rangle=0$

Example 3.1.8:

If u and v are vectors in an inner product space V, find

1. $\langle 2u-7v,3u+5v\rangle$
2. $\langle 3u-4v,5u+v\rangle$ (Homework)

Theorem 3.1.9:

In an inner product space V, for a vector $u\in V$, define,

$$W_u=\{v\in V|\langle u, v\rangle=0\}.$$

Prove that W_u is a subspace of V.

Proof:

$W_u\neq\phi$ (why?)

Let a,b be scalars and $v_1,v_2\in W_u$.

$\langle u, av_1+bv_2\rangle=a\langle u,v_1\rangle+b\langle u,v_2\rangle=a(0)+b(0)=0.$

3.2 Normed vector space

Definition 3.2.1

Let $\langle\ ,\ \rangle$ be an inner product on a vector space V.

1. The norm or the length of $u\in V$ is defined as follows

$$||u||=\sqrt{\langle u,u\rangle}$$

2. The distance between two vectors u and v in V is defined as follows

$$d(u,v)=\|v-u\|$$

The pair $(V, \|.\|)$ is called a normed vector space.

Example 3.2.2:

In **Example 3.1.2**, find

1. $\|x\|$ and $\|x-y\|$
2. $\|(x, y)\|$ and $\|(x_1,y_1),(x_2,y_2)\|$

Example 3.2.3:

In **Example 3.1.4**, find

$\|f\|$ and $\|f-g\|$

Theorem 3.2.4:

Show that $\langle u+v,u-v\rangle=\|u\|^2-\|v\|^2$.

Proof:

$\|u\|^2-\|v\|^2=\langle u,u\rangle-\langle v,v\rangle=\langle u,u\rangle+\langle v,-v\rangle=\langle u+v,u-v\rangle$.

Definition 3.2.5:

In an inner product space V, a vector $u\in V$ is called a unit vector if $\|u\|=1$

Example 3.2.6:

Let a, $b>0$. In a vector space $\Re^2$, define

$\langle (x, y), (x_1,y_1)\rangle=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}$.

1. Prove that $\langle\ ,\ \rangle$ is an inner product on $\Re^2$.(Homework)
2. Show that $\|(x, y)\|=1 \leftrightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Theorem 3.2.7 (Schwarz Inequality):

In an inner product space V,

$$\langle u, v\rangle^2\leq\|u\|^2\|v\|^2$$

Proof:

$u=0 \vee v=0 \to \langle u, v\rangle=0 \wedge (\|u\|=0 \vee \|v\|=0) \to 0\le 0$

Let $u\neq 0\neq v$

$0\le \| xu+v \|^2=\langle xu+v, xu+v\rangle=x\langle u, xu+v\rangle+\langle v, xu+v\rangle$

$=x\big(x\langle u,u\rangle+\langle u,v\rangle\big)+x\langle v, u\rangle+\langle v, v\rangle=\langle u,u\rangle x^2+\langle u,v\rangle 2x+\langle v,v\rangle$

$=\|u\|^2x^2+2\langle u,v\rangle x+\|v\|^2 \to 4\langle u,v\rangle^2\le 4\|u\|^2\|v\|^2 \to \langle u,v\rangle^2\le \|u\|^2\|v\|^2$

Example 3.2.8:

Apply **Schwarz Inequality** in **Example 3.1.4**.

Theorem 3.2.9:

Let $\langle\ ,\ \rangle$ be an inner product over V and $u, v\in V$.

1. $\|u\|\ge 0$.
2. $\|u\|=0 \leftrightarrow u=0$
3. $\|\lambda u\|=|\lambda|\ \|u\|$
4. $\|u+v\|\le\|u\|+\|v\|$ (**triangle inequality**)

Proof:

1. Straightforward.
2. Follows from **Theorem 3.1.7**, Task 4.
3. $\|\lambda u\|^2=\langle \lambda u,\lambda u\rangle=\lambda^2\langle u,u\rangle \to \|\lambda u\|=|\lambda|\ \|u\|$
4. $\|u+v\|^2=\langle u+v,u+v\rangle=\langle u, u+v\rangle+\langle v, u+v\rangle=\langle u, u\rangle+\langle u, v\rangle+\langle v, u\rangle+\langle v, v\rangle$

   $=\|u\|^2+\langle u,v\rangle+\langle u,v\rangle+\|v\|^2==\|u\|^2+2\langle u,v\rangle+\|v\|^2\le\|u\|^2+2\|u\|\ \|v\|+\|v\|^2=(\|u\|+\|v\|)^2$

   $\|u+v\|^2\le(\|u\|+\|v\|)^2 \to \|u+v\|\le\|u\|+\|v\|$

Theorem 3.2.10:

If $\langle\ ,\ \rangle$ is an inner product on V and $u_1,\ldots,u_n$ is a spanning of V, then for each $v\in V$,

$$\langle v,u_i\rangle=0,\ \forall i=1,\ldots, n \to v=0.$$

Proof:

$v=a_1u_1+\ldots+a_nu_n$, for some scalars a_i, $i=1,\ldots,n$.

$\rightarrow\langle v,v\rangle=\langle v,a_1u_1+\ldots+a_nu_n\rangle\rightarrow\langle v,v\rangle=a_1\langle v,u_1\rangle+a_2\langle v,u_2\rangle+\ldots+ a_n\langle v,u_n\rangle=a_1(0)+\ldots+ a_n(0)=0\rightarrow v=0$.

Theorem 3.2.11:

Let V be an inner product space V and u, v be vectors in V.

1. $d(u,v)\geq 0$.
2. $d(u,v)=0\leftrightarrow u=v$
3. $d(u,v)=d(v,u)$
4. $d(u,v)\leq d(u,w)+ d(w,v)$

Proof:

1. Follows form **Theorem 3.2.9, Task 1.**
2. Follows form **Theorem 3.2.9, Task 2.**
3. $d(u,v)= \|u-v\|=\sqrt{\langle u-v,u-v\rangle}=\sqrt{\langle u,u\rangle-2\langle u,v\rangle+\langle v,v\rangle}$

   $d(v,u)= \|v-u\|=\sqrt{\langle v-u,v-u\rangle}=\sqrt{\langle v,v\rangle-2\langle u,v\rangle+\langle u,u\rangle}$
4. $d(u,v)= \|u-v\|=\|(u-w)+(w-v)\|\leq \|u-w\|+\|w-v\|\leq d(u,w)+ d(w,v)$

Exercise 3.2.12:

i. In an inner product space V, for $u,v\in V$, prove each of the following
   1. $\|u+v\|^2=\|u\|^2+2\langle u,v\rangle+\|v\|^2$
   2. $\|u-v\|^2=\|u\|^2-2\langle u,v\rangle+\|v\|^2$
   3. $\langle u,v\rangle=0.25(\|u+v\|^2-\|u-v\|^2)$

ii. In an inner product space V, let $\|u\|=1$, $\|v\|=2$ and $\|w\|=\sqrt{3}$, $\langle u,v\rangle=-1$, $\langle u,w\rangle=0$ and $\langle v,w\rangle=3$. Compute
   1. $\langle v+w,2u-v\rangle$
   2. $\langle u-2v-w,3w-v\rangle$

iii. Let $T:V\rightarrow V$ be an isomorphism of the inner product space V. Show that $\langle u,v\rangle^*=\langle T(u), T(v)\rangle$ is an inner product space on V.

3.3 Orthogonality

Definition 3.3.1:

Let V be an inner product space and $u, v \in V$. We say that u and v are orthogonal if

$$\langle u,v \rangle=0$$

Definition 3.3.2:

Let V be an inner product space. A set $\{u_1,\ldots,u_n\}$ of vectors in V is called orthogonal if

1. $u_i \neq 0$, $\forall i=1,\ldots,n$
2. The set of vectors is pairwise orthogonal, that is $\langle u_i,u_j \rangle$ is orthogonal $\forall i \neq j$.

Additionally, if $\|u_i\|=1$, $\forall i$, then the set $\{u_1,\ldots,u_n\}$ of vectors is called orthonormal.

Example 3.3.3:

1. (-1,3) and (3,1) are orthogonal with respect to **Example 3.1.2**, **Task 2**.
2. $\sin x$ and $\cos x$ are orthogonal in $C[-\pi,\pi]$.
3. (5,2,-3) and (4,-1,6) are orthogonal with respect to **Exercise 3.1.6**, **Task 1**.

Theorem 3.3.4 (The Pythagorean Theorem):

If $\{u_1,\ldots,u_n\}$ is an orthogonal set of vectors, then

$$\left\| \sum_{i=1}^{n} u_i \right\|^2 = \sum_{i=1}^{n} \|u_i\|^2.$$

Proof:

$$\|u_1+\ldots+u_n\|^2=\langle u_1+\ldots+u_n,u_1+\ldots+u_n\rangle=\langle u_1,u_1\rangle+\langle u_2,u_2\rangle+\ldots+\langle u_n,u_n\rangle+\sum_{i\neq j}\langle u_i,u_j\rangle$$

$$=\|u_1\|^2+\ldots+\|u_n\|^2+0=\|u_1\|^2+\ldots+\|u_n\|^2$$

Theorem 3.3.5:

Let $\{u_1,\ldots, u_n\}$ be orthogonal set of vectors, then

i. $\{\lambda_1 u_1,\ldots, \lambda_n u_n\}$ is orthogonal for every $\lambda_i \neq 0$.

ii. $\{\hat{u}_1, \ldots, \hat{u}_n\}$ is orthonormal.

Proof:

i.

1. For each i=1,..,n, $u_i \neq 0 \rightarrow \lambda u_i \neq 0$.
2. $\langle \lambda u_i, \lambda u_j \rangle = \lambda \langle u_i, u_j \rangle = \lambda(0) = 0$.

ii. Homework.

Theorem 3.3.6:

Every orthogonal set of vectors is linearly independent.

Example 3.3.7:

In **Example 3.1.5**, prove that

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$$

is an orthogonal basis of $\Re^3$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 3.3.8:

1. In an inner product space V, prove or disprove:
   - i. $u,v \in V$ are orthogonal $\leftrightarrow \|u+v\| = \|u-v\|$.
   - ii. $\{u,v\}$ is an orthogonal set $\leftrightarrow \|u\| = \|v\|$.
2. In **Example 3.1.5**, Verify whether

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix} \right\}$$

is an orthogonal basis of $\Re^3$or not, where

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

3.4 Orthogonal projections

Definition 3.4.1:

Let W be a subspace of an inner product space V. The orthogonal complement $W^{\perp}$ of W is defined as follows

$$W^{\perp}=\{v\in V|u\in W\rightarrow\langle u, v\rangle=0\}$$

Theorem 3.4.2:

In an inner product space V, the orthogonal complement $W^{\perp}$ of a subspace W of V is a subspace of V.

Proof:

$u\in W\rightarrow\langle u, 0\rangle=0\rightarrow 0\in W^{\perp}$.

Let $v_1,v_2\in W^{\perp}$, a,b be scalars and $u\in W$.

$\langle u, av_1+bv_2\rangle=\langle u,av_1\rangle+\langle u,bv_2\rangle=a\langle u, v_1\rangle+b\langle u,v_2\rangle=a(0)+b(0)=0\rightarrow av_1+bv_2\in W^{\perp}$.

Exercise 3.4.2:

Let U and W be a subspace of a vector space V. Define

$$U\oplus W=\{u+w|u\in U \wedge w\in W\}$$

Show that $U\oplus W$ is a subspace of V.

Definition 3.4.3:

A vector space V is called a direct sum of subspaces U and W if

1. $U\cap W=\{0\}$
2. $V= U\oplus W$.

Exercise 3.4.4:

Let V be a finite dimensional direct sum vector space of subspaces U and W. Then

$$dim\,(V)=dim(U)+dim(W).$$

Theorem 3.4.5:

Let W be a finite dimensional subspace of an inner product space V, then $V=W\oplus W^{\perp}$.

Proof:

We will apply **Definition 3.4.3**,

1. Let $v\in W\cap W^{\perp}$. Since $\langle v,v\rangle=0$, then $v=0$.
2. Clearly, $W\oplus W^{\perp}\subseteq V$.

   Let $v\in V\rightarrow v=0+v$. According to **Theorem 3.1.7**, **Task 3**, $\langle v,0\rangle=0$, then $v\in W^{\perp}$.

Theorem 3.4.6:

Let V be an inner product space, U be the orthogonal complement of W. Define a function T:$V\rightarrow V$ as follows

$$T(v)=u, \text{ where } v=u+w,\ u\in U,\ w\in W.$$

Then

1. T is a linear operator on V. (T is called the projection on U with kernel W)
2. $img(T)=U$
3. $ker(T)=W$