Linear Algebra

Semester I

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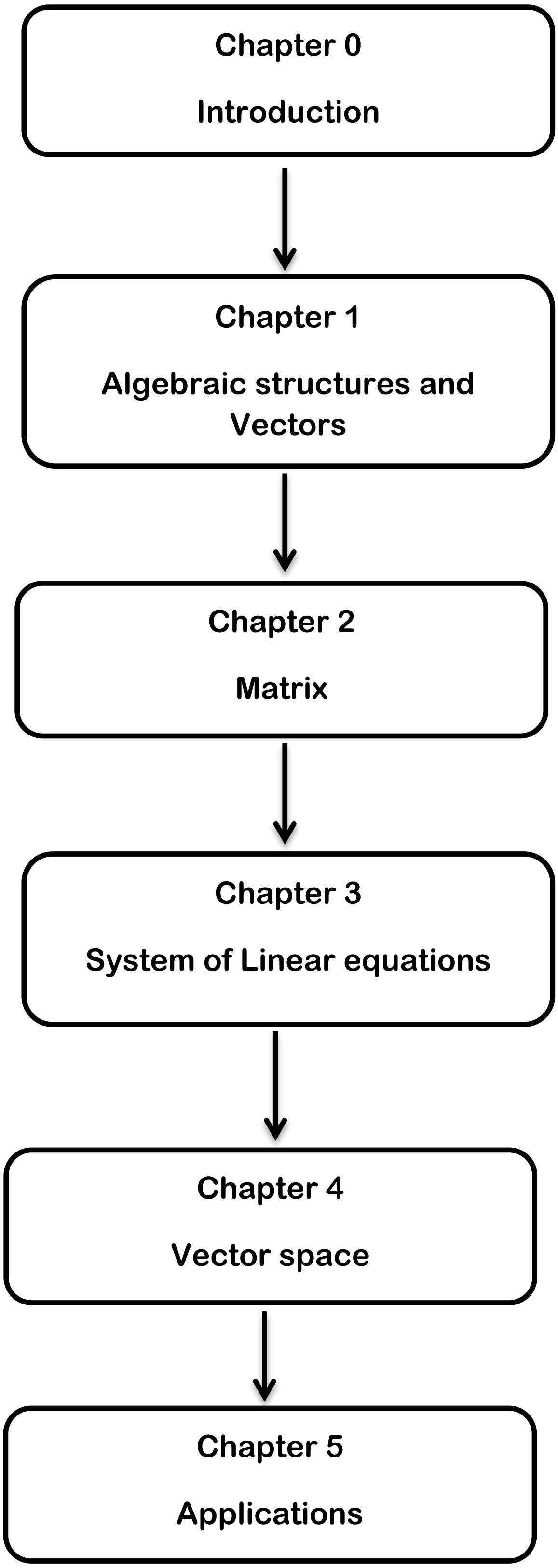
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Chapter 0: Introduction

Definition 0.1:

In mathematics, a linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the results.

For example, from the set $V=\{v_1,\dots v_n\}$,

$$\sum_{i=1}^{n} c_i v_i$$

is a linear combination V.

Definition 0.2:

- **Linear algebra is the study of linear combinations.**
- **It is the study of vector spaces, lines and planes, and some mappings that are required to perform the linear transformations. It includes vectors, matrices and linear functions.**
- **It is the study of linear sets of equations and its transformation properties.**

Chapter 1: Algebraic structures and Vectors

1.1 Algebraic structure

Definition 1.1.1:

Let $M\neq\phi$. A binary operation on the set M is a mapping $*$ from $M\times M$ to M. In other words if the following implication holds

$$a,\ b\in M\rightarrow *\big((a,b)\big)\in M$$

or

$$a,\ b\in M\rightarrow a*b\in M \text{ (common)}$$

In this case, the pair $(M,*)$ is called an algebraic structure. Notice that an algebraic structure can be a set with one, two or more binary operations.

Examples 1.1.2:

1. **The standard addition + is a binary operation on N.**
2. **The standard multiplication • is a binary operation on Z.**
3. **The standard subtraction – is not a binary operation on N.**
4. **The standard division / is not a binary operation on R.**
5. **The standard division / is a binary operation on $R\backslash\{0\}$.**
6. **The union is a binary operation on *Power* (X), for a set X.**
7. **The intersection is a binary operation on *Power* (X), for a set X.**
8. **The difference is a binary operation on *Power* (X), for a set X.**

Definition 1.1.3:

Let $*$ be a binary operation on the set M,

1. **$*$ is called associative on M, if the following implication holds**

   $a, b, c\in M\rightarrow(a*b)*c=a*(b*c)$
2. **$*$ is called commutative on M, if the following implication holds**

   $a, b\in M\rightarrow a*b=b*a$
3. **$e\in M$ is called the identity of M with respect to $*$, if the following implication holds**

   $a\in M\rightarrow a*e=a=e*a$
4. **For $a\in M$, the element $a^{-1}\in M$ is called the inverse of M with respect to $*$, if the following implication holds**

   $a*a^{-1}=a^{-1}*a=e$

Definition 1.1.4:

Let $*$ and o be two binary operations on the set M, we say that

1. **o is distributed on $*$ from the left hand side, if the following implication holds**

   $a, b, c\in M\rightarrow a\text{o}(b*c)= (a\text{o}b)*(a\text{o}c)$
2. **o is distributed on $*$ from the right hand side, if the following implication holds**

   $a, b, c\in M\rightarrow (b*c)\text{o}a=(b\text{o}a)*(c\text{o}a)$
3. **o is distributed on $*$, if it is distributed from left and right hand sides.**

Examples 1.1.5:

1. **The standard addition + is associative and commutative on *Z*.**
2. **The standard subtraction is neither associative nor commutative on *Z*, since $2,3,4 \in Z \rightarrow 2-(3-4)=3 \neq -5=(2-3)-4$**
3. **ϕ is the identity element of *Power* (*X*) with respect to $\cup$. While *X* is the identity element of *Power* (*X*) with respect to $\cap$.**
4. **In the algebraic structure (*Power* (*X*),$\cup$), ϕ is the only element that has inverse. While in (*Power* (*X*),$\cap$), *X* is the only element that has inverse.**
5. **The standard multiplication $\bullet$ is distributed on the standard addition + in *R*.**

Remark 1.1.6:

Henceforth, we write *ab* instead of $a\bullet b$.

Example 1.1.7:

Consider *R*. Define $*$ on *R* as follows

$a*b=a+b+1$

Exercise 1.1.8:

Consider *R*. Define $*$ on *R* as follows $a*b=\frac{ab}{4}$

1. **Verify whether $*$ is associative or commutative.**
2. **Does *R* have the identity element?**
3. **Does any $a \in R$ have the inverse?**

Definition 1.1.9:

Let $F \neq \phi$ and $+_F$, $\bullet_F$ be two binary operations on *F*. A triple $(F,+_F,\bullet_F)$ is called a field if the following conditions hold

1. **$+_F$ is associative on *F*.**
2. ***F* has the identity element with respect to $+_F$, we use 0 to this identity element.**
3. **Any element of $a \in F$ has the inverse with respect to $+_F$. The inverse of any element $a \in F$ is denoted by $-_F a$.**

4. $+_F$ is commutative on F.
5. $\bullet_F$ is associative on $F\backslash\{0\}$.
6. $F\backslash\{0\}$ has the identity element with respect to $\bullet_F$, we use e to this identity element.
7. Any element of $F\backslash\{0\}$ has the inverse with respect to $\bullet_F$. The inverse of any element $a\in F\backslash\{0\}$ is denoted by a^{-1}.
8. $\bullet_F$ is commutative on $F\backslash\{0\}$.
9. $\bullet_F$ is distributed on $+_F$.

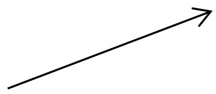
Example 1.1.10:

1. $(Q,+,\bullet)$ is afield.
2. $(R,+,\bullet)$ is a field.
3. $(Z,+,\bullet)$ is not a field.

1.2 Vectors

Definition 1.2.1:

A vector is a quantity that has both magnitude and direction.



Vectors describe the movement of an object with respect to another point.

Velocity, acceleration and force are some examples of vectors.

- Speed is the time rate at which an object is moving along a path, while velocity is the rate and direction of an object's movement.
- Acceleration is the rate of change of the velocity of an object with respect to time.
- A force is an influence that can change the motion of an object.

In R^2, any vector can be written as $\vec{v}=(v_x,v_y)$, where v_x and v_y are real numbers.

$|v|= r$ (magnitude)$=\sqrt{v_x^2+v_y^2}$

Direction $(\theta)=\tan^{-1}(\frac{v_y}{v_x})$

Notice that (r, θ) is the polar representation of v.

Example 1.2.2:

The vector v=(1,$\sqrt{3}$) has magnitude r=$\sqrt{1^2+(\sqrt{3})^2}$=2 and direction θ=tan$^{-1}(\frac{\sqrt{3}}{1})=\frac{\pi}{3}$ or θ=π+$\frac{\pi}{3}$. According to the position of v, θ=$\frac{\pi}{3}$.

So (2, $\frac{\pi}{3}$) is the polar representation of v.

On the other hand, for a point in polar coordinate (2, $\frac{\pi}{3}$),

x=$r$cos($\frac{\pi}{3}$)=2($\frac{1}{2}$)=1 and y=r sin($\frac{\pi}{3}$)=2($\frac{\sqrt{3}}{2}$)= $\sqrt{3}$.

Hence (1,$\sqrt{3}$) is the Cartesian representation of (2, $\frac{\pi}{3}$).

Definition 1.2.3:

1- **For any vector $\vec{v}$, the -$\vec{v}$ is the vector with the same magnitude and opposite direction.**
2- **The zero or the null vector is a vector that has a zero magnitude and no direction and denoted by $\vec{0}$.**
   Pulling a rope from its two ends with equal force but in opposite directions is an example of null force (zero vector).

Definition 1.2.4:

Let $\vec{v}$=(v_x,v_y) and $\vec{w}$=(w_x,w_y) be two vectors in R^2 and c (scalar)∈R. The vector addition and the scalar multiplication of vectors are defined as follows

1. **$\vec{v}+\vec{w}$=(v_x+w_x,v_y,+w_y). (vector addition)**
2. **c$\vec{v}$=(cv_x,cv_y). (scalar multiplication)**

Definition 1.2.5:

A vector $\vec{v}$=(v_x, v_y) is called a unit vector if |$\vec{v}$|=1.

The standard unit vectors in R^2 are $\hat{\imath}$=(1,0) and $\hat{\jmath}$=(0,1).

Any vector can be written as $\vec{v}$= $v_x\hat{\imath}$+ $v_y\hat{\jmath}$.

Proposition 1.2.6:

For any vector $\vec{v}$, the vector $\vec{w}$=$\frac{\vec{v}}{|v|}$ is a unit vector.

Remark 1.2.7:

We may define vectors in R^n as follows

$\vec{v}$=(v_1,…,v_n), where $v_i \in R$, for all i.

In this case, $|\vec{v}|$=$\sqrt{\sum_{i=1}^{n} v_i^2}$.

Chapter 2: Matrix Theory

2.1 Basic definitions

Definition 2.1.1:

A matrix *A* is a rectangular array of numbers arranged into rows and columns.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$

- ***m* is the number of rows and *n* is the number of columns.**
- **The object a_{ij} is the entry of the matrix *A*, located in *i*-th row and *j*-th column.**
- ***A* is called an *m*×*n* matrix or *A* is a matrix of dimension (order) *m*×*n*.**

Definition 2.1.2:

A matrix of the dimension

1. **1×*n* is called a row vector.**
2. ***m*×1 is called a column vector.**

According to Definition 2.1.2, any vector $\vec{v}$ in R^n can be considered as a 1×*n* matrix

$$\vec{v}=(v_1 \quad \cdots \quad v_n),$$

or an *n*×1 matrix

$$\vec{v}=\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Definition 2.1.3:

Let *A* be a matrix of dimension *m*×*n*, *A* is said to be

1. **Square, if *m*=*n*.**
2. **Zero matrix "0", if a_{ij}=0, for all *i, j*.**

   Notice that 0 matrix can be of any dimension.

Definition 2.1.4:

Let *A* be a square matrix of dimension *n*,

1. **The set of entries a_{ii}, is called the main diagonal of *A*.**
2. **The sum $\sum_{i=1}^{n} a_{ii}$ is called the *trace* of the matrix *A*, and is denoted by *trac*(*A*).**
3. ***A* is called a diagonal matrix if**

$$i \neq j \rightarrow a_{ij}=0$$

4. ***A* is called identity matrix, if**

$$a_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

Usually, identity matrix of order *n* is denoted by I_n. Notice that, the identity matrix can be defined by the Kronecker delta function δ_{ij}.

Definition 2.1.5:

Let *A* be an $m \times n$ matrix, the transpose of *A* is the matrix generated by interchanging the rows and columns of the matrix *A* and denoted by A^T. Clearly A^T is a matrix of dimension $n \times m$ matrix.

$$a_{ij} \text{ is an entry in } A \leftrightarrow a_{ji} \text{ is an entry in } A^T.$$

Example 2.1.6:

Let $A=\begin{pmatrix} 2 & 0 & -3 \\ 5 & \sqrt{3} & 4 \end{pmatrix}$**, then** $A^T=\begin{pmatrix} 2 & 5 \\ 0 & \sqrt{3} \\ -3 & 4 \end{pmatrix}$.

Exercise 2.1.7:

Let *A* be a square matrix of order *n*, then

$$trac\,(A^T)=trac(A).$$

2.2 Matrix operations

2.2.1 Addition of matrices

Definition 2.2.1.1:

Let $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}$ be two matrices of dimension $m\times n$, then the sum of *A* and *B* is given by

$$A+B=(a_{ij}+b_{ij})$$

This means that $A+B=(c_{ij})$, where $c_{ij} = a_{ij} + b_{ij}$.

Example 2.2.1.2:

Let $A=\begin{pmatrix} 5 & 2 & -4 \\ 2 & -1 & 1 \\ 1 & 0 & 7 \\ 8 & 4 & 6 \end{pmatrix}$ **and** $B=\begin{pmatrix} 4 & 0 & 2 \\ -7 & 1 & 0 \\ 2 & 9 & 3 \\ 4 & -1 & 11 \end{pmatrix}$

Exercise 2.2.1.3:

Prove that, for any matrix A of dimension $m\times n$,

1. **$A+0=A=0+A$, where 0 is the $m\times n$ zero dimensional matrix.**
2. **$(A^T)^T = A$.**

Theorem 2.2.1.4:

Let A, B and C be matrices of order $m\times n$, then

1. **$A+B$ is a matrix of order $m\times n$.**
2. **$A+(B+C)=(A+B)+C$.**
3. **$A+B=B+A$**
4. **$(A+B)^T=A^T+B^T$**
5. **$trac(A+B)=trac(A)+trac(B)$, where $m=n$.**

2.2.2 Multiplication of matrices by a scalar

Definition 2.2.2.1:

Let $A=(a_{ij})_{m\times n}$ and λ be a scalar, then the multiplication λA is a matrix of dimension $m\times n$ and is given by

$$\lambda A=(\lambda a_{ij})_{m\times n}$$

It is easy to verify that $\lambda A=A\lambda$.

Example 2.2.2.2:

Let A be the matrix given in Example 2.2.1.2 and $\lambda=2$.

Remark 2.2.2.3:

From Definition 2.2.1.1 and Definition 2.2.2.1, we may define the subtraction of two matrices of the same order (dimension).

Definition 2.2.2.4:

Let $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}$ be two matrices of dimension $m\times n$, then the subtraction of A and B is denoted by A-B and is given by

$$A-B=A+(-B)$$

Exercise 2.2.2.5:

Let A and B be matrices of the same order and $\lambda\in R$, then

1. $\lambda A=A\lambda$
2. $\lambda(A+B)=\lambda A+\lambda B$
3. **Let A be a matrix of order $m\times n$, then, $A-A=0$.**
4. $(\lambda A)^T=\lambda A^T$.
5. $A0=0=0A$.
6. $trac(\lambda A)=\lambda trac(A)$.

2.2.3 Multiplication of two matrices

Definition 2.2.3.1:

Let $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{n\times k}$ be two matrices. The multiplication of A and B is a matrix $C=AB$ of order $m\times k$ such that

$$c_{ij}=\sum_{u=1}^{n} a_{iu}b_{uj}$$

Example 2.2.3.2:

$A=\begin{pmatrix} 2 & -1 & 0 \\ 1 & 4 & 5 \end{pmatrix}$ **and** $B=\begin{pmatrix} 3 & -2 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ -4 & 6 & -1 & -1 \end{pmatrix}$

Theorem 2.2.3.3:

Let *A*, *B* and *C* be matrices of order $k\times l$, $l\times m$ and $m\times n$ respectively, then

1. $A(BC)=(AB)C$.
2. $\lambda(AB)=(\lambda A)B=A(\lambda B)$.
3. $(AB)^T=B^TA^T$.

Theorem 2.2.3.4:

Let *A* and *B* be square matrices of the same order, then

$$trac(AB)= trac(BA).$$

Theorem 2.2.3.5:

Let $A=(a_{ij})$, $B=(b_{ij})$ and $C=(c_{ij})$ be conformable matrices, then

1. $C(A+B)= CA+CB$.
2. $(A+B)C=AC+BC$.

Theorem 2.2.3.6:

Let *A* be a square matrix of order *n*, then

$$AI_n=A=I_nA.$$

Exercise 2.2.3.7:

Give an example to show that $AB\neq BA$, even *A* and *B* are square matrices of the same order.

Remark 2.2.3.8:

Let *A* be a matrix and *n* be a positive integer,

$$A^n=\underbrace{AA\ldots A}_{n\text{-times}}$$

2.3 Determinant of matrix

- **Determinant of square matrix of order two**

Definition 2.3.1:

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$**. The determinant of** A **is defined as follows**

$$|A| = det\,(A) = a_{11}a_{22} - a_{12}a_{21}$$

Example 2.3.2:

Let $A = \begin{pmatrix} -3 & 2 \\ -4 & 5 \end{pmatrix}$

- **Determinant of square matrix of order** n**,** $n>2$**.**

Definition 2.3.3:

Let $A=(a_{ij})$ **be a square matrix of order** n**,** $n>2$

1. **The minor matrix of** A **is the matrix** $M=(m_{ij})$ **such that** m_{ij} **is the determinant of the remaining matrix after eliminating the** ***i-th*** **row and the** ***j-th*** **column form the original matrix.**
2. **The cofactor matrix of** A **is the matrix** $C=(c_{ij})$**, where** $c_{ij}=(-1)^{i+j}m_{ij}$**.**

Definition 2.3.4:

Let $A=(a_{ij})$ **be a square matrix of order** n**,** $n>2$**. The determinant of** A **is defined as follows,**

$$|A| = \sum_{j=1}^{n} a_{ij}\, c_{ij}$$

Notice that the above formula can be applied for square matrices of order 2 as well.

Example 2.3.5:

Let $A=\begin{pmatrix} 2 & -3 & 1 \\ 5 & -1 & 0 \\ 4 & 7 & 3 \end{pmatrix}$

Exercise 2.3.6:

Find the determinant of the matrix

1. $A=\begin{pmatrix} -1 & 4 & 0 \\ 8 & 4 & 2 \\ -3 & 1 & 5 \end{pmatrix}$
2. $A=\begin{pmatrix} 3 & 2 & -1 & 6 \\ 0 & 5 & 2 & 2 \\ 9 & 8 & 1 & -4 \\ 4 & -2 & 1 & 7 \end{pmatrix}$

Theorem 2.3.7 (Properties of determinant of a matrix)

Let A and B be square matrices of order n, then

1. $|AB|=|A||B|$.
2. $|I_n|=1$.
3. **The sign of the determinant changes under the row interchange.**
4. **The sign of the determinant changes under the column interchange.**
5. **If all the elements of a row (or column) are zeros, then the value of the determinant is zero.**
6. $|\lambda A|=\lambda^n|A|$.
7. **Determinant is a linear function of a row or a column.**
8. **If two rows (columns) are identical, then $|A|=0$.**
9. $|A|=|A^T|$.
10. **Is $|A+B|=|A|+|B|$?**

Definition 2.3.8:

A square matrix A is said to be non-singular if $|A|\neq 0$. Otherwise it is called singular.

2.4 Invertible matrix

Definition 2.4.1:

Let A be a non-singular matrix of order n, a non-singular square matrix B of order n is called the inverse of A if

$$AB=I_n=BA$$

If such a matrix *B* exists satisfies the above formula, then we say that *A* is an invertible matrix of order *n*. Notice that, if *B* is the inverse of *A*, then *A* is the inverse of *B* as well. We use the notation A^{-1} to the inverse of a matrix *A*.

Definition 2.4.2:

Let *A* be a square matrix of order *n*, The adjoint of the matrix *A*, denoted by *adj*(*A*) is the transpose of the cofactor matrix of *A*. That means,

$$adj(A)=C^T$$

Theorem 2.4.3:

Let *A* be a non-singular matrix of order *n*, the inverse of *A* can be find from the following formula:

$$A^{-1}=\frac{1}{|A|}adj(A)$$

Example 2.4.4:

Let $A=\begin{pmatrix} 2 & -3 & 1 \\ 5 & -1 & 0 \\ 4 & 7 & 3 \end{pmatrix}$

Theorem 2.4.5:

Suppose that *A* is an invertible matrix of order *n*. Prove the following

1. $|A^{-1}|=\frac{1}{|A|}$
2. $(A^{-1})^{-1}=A$
3. $(AB)^{-1}=B^{-1}A^{-1}$
4. $(A^T)^{-1}=(A^{-1})^T$
5. $(\lambda A)^{-1}=\frac{1}{\lambda}A^{-1}$
6. $(A^{-1})^n=A^{-n}$, **where** $A^{-n}=\frac{1}{A^n}$

1.5 Some types of square matrix

Definition 2.5.1:

Let A=(a_{ij}) be a square matrix of order n, A is said to be

1. **Symmetric, if A^T=A.**
2. **Skew symmetric if A^T=-A.**
3. **Upper triangular if a_{ij}=0, for all i>j.**
4. **Lower triangular if a_{ij}=0, for all i<j.**
5. **Triangular, if it is upper triangular or lower triangular or both.**

1.6 Rank of matrix

Definition 2.6.1:

Let A be an m×n matrix (possibly m=n). The rank of A is the order of the nonzero determinant of highest order that may be formed from the elements of a matrix by selecting arbitrarily an equal number of rows and columns.

Example 2.6.2:

Let $A=\begin{pmatrix} 2 & -3 & 1 & -4 \\ 5 & -2 & 3 & 2 \\ -4 & 6 & -2 & 8 \end{pmatrix}$ **and** $B=\begin{pmatrix} 2 & -3 & 6 & -4 \\ 5 & -2 & 15 & 2 \\ -4 & 6 & -12 & 9 \end{pmatrix}$

1.7 Row echelon form matrix

Definition 2.7.1:

The leading entry in each entire row of a matrix is considered as the first nonzero entry in that row.

Definition 2.7.2:

A matrix is said to be in row echelon form *ref* (echelon form), if

1. **Each leading entry is in a column to the right of the leading entry in the previous row.**
2. **Rows with all zero elements, if any, are below rows having nonzero elements.**

Example 2.7.3:

The following matrices are in *ref*

$$A=\begin{pmatrix} 1 & 5 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C=\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

- **Pivot position and pivot column**

Definition 2.7.4:

A pivot position in a matrix is the location of a leading entry in the *ref* of a matrix. A pivot column is a column that contains a pivot position.

Definition 2.7.5:

A matrix is said to be in reduced row echelon form *rref* (canonical form), if

1. **The matrix satisfies conditions for a *ref*.**
2. **The leading entry in each row is the only nonzero entry in its column.**

Example 2.7.6:

The following matrices are in *rref*.

$$A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise 2.7.7

Transform the following matrices to *ref*

1. $A=\begin{pmatrix} 4 & 3 & -1 & 2 \\ 3 & 3 & -2 & 6 \\ 5 & 1 & 1 & -2 \end{pmatrix}$
2. $B=\begin{pmatrix} 5 & 6 & 3 \\ -4 & 0 & 1 \\ -1 & 3 & 2 \\ 2 & 4 & -3 \end{pmatrix}$

Chapter 3: System of Linear Equations

3.1 Basic definitions

Definition 3.1.1:

A system of *m* linear equations in *n* unknown can be defined as follows,

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix} \quad \ldots (3.1)$$

where a_{ij} and b_i are scalars in F.

A solution of system (3.1) is a vector $(s_1 \cdots s_n) \in F^n$ which satisfies all *m* equations simultaneously.

$$\begin{matrix} a_{11}s_1 + \cdots + a_{1n}s_n = b_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}s_1 + \cdots + a_{mn}s_n = b_m \end{matrix}$$

Definition 3.1.2:

The Linear System (3.1) is called homogeneous, if b_i=0, for all *i*. Otherwise is called nonhomogeneous.

Definition 3.1.3:

Two linear systems

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix}$$

and

$$\begin{matrix} c_{11}x_1 + \cdots + a_{1n}x_n = d_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{k1}x_1 + \cdots + a_{kn}x_n = d_k \end{matrix}$$

are called equivalent if they both have exactly the same solutions.

Definition 3.1.4: (Row Elementary Operations-REO)

The following operations on a system of linear equations are called row elementary operations.

Type I: Interchange the rows r_i and r_k.

Type II: Multiply the row r_i by a nonzero scalar λ.

Type III: Replace the row r_i by the row $\lambda r_k + r_i$, $i \neq k$.

Theorem 3.1.5:

For a linear system, if we apply at least (one or more) elementary operations of REO, then we obtain an equivalent system to the original one.

2.2 Representing linear systems with matrices

System (3.1) can be written as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Or, in matrix equation, as

$$AX = B \quad \dots \ (3.2)$$

where,

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ **is called the coefficients matrix.**

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ **is called the column matrix of unknowns (variables).**

$B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ **is called the column matrix of constants.**

Clearly, any solution of (3.2) is a solution of (3.1) and vice versa.

Theorem 3.2.1:

In Definition 3.1.1, if F is infinite and System (3.1) has more than one solution, then there is infinite number of solutions.

Proof:

Let X_1 and X_2 be two distinct solutions of (3.2). Then $X^*=X_1+\lambda(X_1-X_2)$, $\lambda\in F$ is also a solution of (3.2).

Definition 3.2.2:

The Augmented matrix of the system (3.2) is defined by

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

Definition 3.2.3:

A linear equation (hyperplane)

$$a_1x_1+\cdots+a_nx_n=b \ldots (3.3)$$

is said to be degenerate, if a_i=0, for all i. Otherwise is called a non-degenerate linear equation.

Remark 3.2.4:

Let System (3.1) contains a degenerate equation,

1. **If $b\neq0$, then the system has no solution.**
2. **If $b=0$, then the degenerate equation may be deleted from the system without changing the solution of the system.**

Definition 3.2.5:

For a non-degenerate equation, the leading unknown (variable) is the variable with the first non-zero coefficient (entry).

Definition 3.2.6:

A matrix *A* is said to be row equivalent to a matrix *B*, if *B* can be obtained by a finite sequence of REO.

Theorem 3.2.7:

Any matrix is a row equivalent to a matrix in *ref*.

Proof: Not required.

Theorem 3.2.8:

Any matrix is a row equivalent to a unique matrix in *rref*.

Proof: Not required.

Theorem 3.2.9:

Let $AX=B$ and $CX=D$ be two linear systems of the same number of equations and unknowns. If the augmented matrix $[A:B]$ and $[C:D]$ are row equivalents, then both systems are equivalents.

3.3 Solving linear systems

Definition 3.3.1:

In a system of *ref*, an unknown x_i is called a basic variable if it corresponds to a pivot column *i*. Otherwise x_i is called a free variable.

Example 3.3.2:

The solution of the following system

$$\begin{gathered} 2x-y+z=1 \\ x+y-2z=4 \end{gathered}$$

is defined by

$$\begin{gathered} x=\frac{5}{3}+\frac{1}{3}t \\ y=\frac{7}{3}+\frac{5}{3}t \\ z=t \end{gathered}$$

Definition 3.3.3:

The Linear System (3.1) is said be

1. **Consistent, if it has a solution (either unique or infinite number of solutions).**
2. **Inconsistent, if it has no solution.**

Theorem 3.3.4:

Consider a system of *ref* with *m* equations and *n* unknowns. Let *r* be the number of nonzero rows. (Clearly $r \le m$)

1. **The system is inconsistent, if among the nonzero rows, there is a row for which all entries are zero except the entry of the last column.**
   That means, a row like $(0\ 0\ \dots\ 0|b)$, $b \ne 0$.
2. **If $r \le n$, then the system is consistent**
   2.1 If $r < n$, then the system has infinite number of solutions.
   2.2 If $r = n$, the system has unique a solution.

3.4 Solving system of linear equations

3.4.1 Gaussian elimination method

Gaussian elimination is a method for solving matrix equation (3.2). The steps are

1. **Compose the augmented matrix**
2. **Perform elementary row operations to put the augmented matrix into the upper triangular form.**
3. **Use backwards substitution to find the values of the unknowns.**

Example 3.4.1.1

Solve the following system by using Gaussian elimination method

1.

$$\begin{gathered} 2x - 3y + 4z = -3 \\ 5x + 2y - z = 5 \\ 6x - 2y + 3z = 2 \end{gathered}$$

2.

$$\begin{gathered} x + 2y - z = 2 \\ 2x + 5y - 3z = 1 \\ x + 4y - 3z = 3 \end{gathered}$$

3.

$$
\begin{aligned}
-2x + 3y + 3z &= -9 \\
3x - 4y + z &= 5 \\
-5x + 7y + 2z &= -14
\end{aligned}
$$

3.4.2 Gauss-Jordan elimination

We may use Gauss-Jordan elimination for solving linear system (3.1). In fact we do the same steps as we have done in Gaussian elimination method as well as we continue until the augmented matrix becomes in reduced row echelon form.

Example 3.4.2.1:

Solve the following system (Use Gauss-Jordan elimination)

$$
\begin{aligned}
2x + 3y + z &= 1 \\
x + 5y - 4z &= -12 \\
4x - y + 2z &= 9
\end{aligned}
$$

Remark 3.4.2.2:

We may use Gauss-Jordan elimination to find the inverse of matrices.

Example 3.4.2.3:

Use Gauss-Jordan elimination to find the inverse of the following matrices

1. $A = \begin{pmatrix} -3 & 5 \\ 2 & 1 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 & 3 & 1 \\ -1 & -2 & 1 \\ 3 & 7 & -1 \end{pmatrix}$

Exercise:

A system *AX=B*, where *A* is a square matrix of order *n* has a unique solution if and only if *A* is invertible.

3.4 Homogeneous linear system of equations

A homogeneous linear system of m equations and n unknowns can be defined as follows,

$$\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array} \quad \ldots (3.4)$$

In fact, System (3.4) has always $\{x_1=0,\ldots, x_n=0\}$ as a solution called the zero or trivial solution.

Theorem 3.5.1:

In the *ref* of System (3.4), let k be the number of nonzero rows.

1. **If $k<n$, then the system has a nontrivial solution.**
2. **If $k=n$, then the system has a unique solution.**

Corollary 3.5.2:

Consider a homogeneous linear system of equations

$$AX=0, \text{ where } A \text{ is an } m\times n \text{ matrix.}$$

If $m<n$, then the system has a nontrivial solution.

Remark 3.5.3:

Let A be a square system in the following homogeneous linear system,

$$AX=0.$$

If $|A|=0$, then the system has a nontrivial solution. Is the converse true? Explain your answer.

Definition 3.6.7:

For a nonhomogeneous system $AX=B$, the system $AX=0$ is called its associated homogenous system.

Theorem 3.6.8:

Let x_p be a particular solution of $AX=B$.

1. **If x_h is a solution of the associated homogeneous system, then x_p+x_h is a solution of $AX=B$.**
2. **If u is a solution of $AX=B$, then there exists a solution x_h of the associated homogeneous system such that $u=x_p+x_h$.**

Exercise 3.6.9:

For each of the following, if it is possible give an example. Otherwise explain why?

1. **An inconsistent system with an associated homogeneous system that has infinitely many solutions.**
2. **An inconsistent system with an associated homogeneous system that has a unique (trivial) solution.**

Chapter 4: Vector space (Linear space)

4.1 Basic definitions

Remark:

Let λ and μ be scalars in a field *F*.

- **Axioms of vector addition (⊕)**

Let $\vec{v}$, $\vec{w}$ and $\vec{z}$ be vectors in a set *V* and ⊕ be an operation

1. **Closure property**

$$\vec{v} \oplus \vec{w} \in V$$

2. **Associative property**

$$\vec{v} \oplus (\vec{w} \oplus \vec{z}\,) = (\vec{v} \oplus \vec{w}\,) \oplus \vec{z}$$

3. **Commutative property**

$$\vec{v} \oplus \vec{w} = \vec{w} \oplus \vec{v}$$

4. **The existence of the identity vector**

$$\vec{v} \oplus \vec{0} = \vec{v} = \vec{0} \oplus \vec{v}$$

5. **The existence of the additive inverse**

$$\vec{v} \oplus (-\vec{v}) = \vec{0} = (-\vec{v}) \oplus \vec{v}$$

- **Axioms of vector multiplication by a scalar (⊙)**

Let $\vec{v}$ and $\vec{w}$ be vectors in *V* and an operation ⊙

6.

$$\lambda \odot \vec{v} \in V$$

7.

$$\lambda \odot (\vec{v} \oplus \vec{w}\,) = (\lambda \odot \vec{v}) \oplus (\lambda \odot \vec{w})$$

8.

$$(\lambda + \mu) \odot \vec{v} = (\lambda \odot \vec{v}) \oplus (\mu \odot \vec{v})$$

9.

$$\lambda \odot (\mu \odot \vec{v}) = (\lambda \cdot \mu) \odot \vec{v}$$

10.

$$\mathbf{1} \odot \vec{v} = \vec{v} = \vec{v} \odot \mathbf{1}$$

Definition 4.1.2:

A vector space (linear space) V over a field F consists of a set of vectors V and scalars in F such that the axioms of vector addition and axioms of vector multiplication by a scalar holds.

Example 4.1.3:

Let $V=\{\vec{v} | \vec{v}$ is a vector in $R^2\}$ and $F=R$. Then V is vector space over R.

Example 4.1.4:

Let $V=\{\vec{v} | \vec{v}$ is a vector in $R^n\}$ and $F=R$. Then V is vector space over R.

Example 4.1.5:

Let $P_2=\{p(x) |\ p(x)=a_2x^2 + a_1x + a_0,\ a_i \in R\}$ be the set of all polynomials of degree less than or equal to 2 together with the zero polynomial.

Then $V=P_2$ is vector space over the field $F=R$ under the addition of polynomials and standard multiplication.

More generally, let

$P_n=\{p(x) |\ p(x)=\sum_{i=0}^{n} a_i x^i,\ a_i \in R\}$ be the set of all polynomials of degree less than or to n together with the zero polynomial.

Then $V=P_n$ is vector space over the field $F=R$.

Example 4.1.6:

Let M be the set of all matrices of order $m\times n$ with real coefficients.

Then M is a vector space under the matrix addition and the matrix multiplication by scalars.

Example 4.1.7:

Let $A\neq\phi$ and $R^A=\{f | f:A\to R$ is a mapping$\}$.

Define + on R^A such that

$$(f+g)_{(x)}=f\,(x)+g\,(x)$$

and the scalar multiplication is the standard multiplication of R.

Then R^A is a vector space over R.

Theorem 4.1.8:

Let V be a vector space over a field F, then

1. The additive identity $\vec{0}$ is unique.
2. The additive inverse $-\vec{v}$ of $\vec{v}$ is unique.
3. $0\vec{v}=\vec{0}$. (0 is the zero in F, while $\vec{0}$ is the zero vector).
4. $(-1)\vec{v}=-\vec{v}$.

Theorem 4.1.9 (Cancellation law)

Let $\vec{v}$, $\vec{w}$ and $\vec{z}$ be vectors in a vector space V over a filed F, then

$$\vec{v}+\vec{w}=\vec{v}+\vec{z}\rightarrow\vec{w}=\vec{z}$$

4.2 Linear combination and span

- **Linear combination**

Definition 4.2.1:

Let V be a vector space over a field F and $\vec{v_1}, \ldots, \vec{v_n}$ be a set of vectors in V. A vector $\vec{w}\in V$ is said to be a linear combination of vectors $\vec{v_1}, \ldots, \vec{v_n}$ if

$\exists$ scalars $\lambda_1, \ldots, \lambda_n$ in F (not necessary to be distinct) such that

$$\vec{w}=\sum_{i=1}^{n}\lambda_i\vec{v}_i$$

Example 4.2.2:

Consider the vector space R^2 over the field R and $\vec{v_1}=(-1,3)$ and $\vec{v_2}=(4,1)$. Check whether (10, 9) is a linear combination of A or not.

Example 4.2.3:

Consider the vector space R^3 over the field R and let $A=\{(0,2,4), (3,5,1), (3,7,5)\}$. Check whether (1,2,-1) is a linear combination of A or not.

- **Span**

Definition 4.2.4:

Let V be a vector space, a subset S={$\vec{v}_1$, …, $\vec{v}_n$}⊆V is said to span (generate) V, if the following implication holds:

$$\vec{v}\in V\rightarrow\vec{v} \text{ is a linear combination the set } S.$$

Example 4.2.5: In the vector space R^4 over the field R, the set

$$S = \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

spans the vector space R^4 over R.

Generally, for the vector space R^n over the field R, the set

$$S = \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$$

spans the vector space R^n over R.

Example 4.2.6:

In the vector space P_3 over the field R, the set

$$S = \{x^3, x^2, x, 1\}$$

spans P_3.

Proposition 4.2.7:

Let S={$\vec{v}_1$, …, $\vec{v}_n$} be a span of a vector space V. For any $\vec{w}\in V$, the set S_1={$\vec{w}$, $\vec{v}_1$, …, $\vec{v}_n$} spans V.

Proposition 4.2.8:

Let S={$\vec{v}_1$, …, $\vec{v}_n$} be a span of a vector space V. If $\vec{v}_k\in S$ is a linear combination of some of the other v_i's. Then $S\backslash\{\vec{v}_k\}$ spans V.

Example 4.2.9:

In the vector space R^3 over the field R, the set

$\boldsymbol{S} = \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \right\}$ **spans the vector space R^3 over R.**

Example 4.2.10:

In the vector space R^3 over the field R, the set

$$\boldsymbol{S} = \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{-1} \\ \mathbf{-3} \\ \mathbf{-3} \end{pmatrix} \right\}$$

does not span the vector space R^3 over R.

Example 4.2.11:

In the vector space P_3 over R,

1. **The set $\{1+x+x^2, 1+2x+3x^2, 1+5x+8x^2\}$ spans P_3.**
2. **The set $\{1+2x, -1+x+2x^2, 3-4x^2\}$ does not span P_3.**

Exercise 4.2.12:

In the vector space P_2 over R, let $p_1(x)=2x^2-x+5$, $p_2(x)=x^2+4x-1$ and $p_3(x)=-3x^2-2x$. Check whether $q\,(x)=x^2-2x+3$ is a linear combination of p_1, p_2 and p_3.

4.3 Subspace

Definition 4.3.1:

Let V be a vector space over a field F. A nonempty subset $W \subseteq V$ is said to be a subspace of V over the field F if W is a vector space with the same vector addition and vector scalar multiplication of V over the field F.

Clearly, any vector space has at least two subspaces, $W=\{0\}$ and $W=V$. $W=\{0\}$ is called the trivial subspace.

Theorem 4.3.2:

Let V be a vector space over a field F. A subset $W \subseteq V$ is a subspace of V over the field F if and only if

1. $\vec{0} \in W$.
2. $\vec{u}, \vec{v} \in W \rightarrow \vec{u} + \vec{v} \in W$
3. $\vec{v} \in W \wedge \lambda \in F \rightarrow \lambda \vec{v} \in W$

Proposition 4.3.3:

Let V be a vector space over a field F. A nonempty subset $W \subseteq V$ is a subspace of V over the field F if and only if the following implication holds

$$\vec{u}, \vec{v} \in W \wedge \lambda, \mu \in F \rightarrow \lambda \vec{u} + \mu \vec{v} \in W$$

Example 4.3.4:

Consider the vector space R^n over the field R. The set of all hyperplanes passes through the origin is a subspace of R^n.

Example 4.3.5:

In the vector space R^2 over R, Check whether the following subsets of R^2 is a subspace or not.

1. $W = \{(x, y) \in R^2 | xy = 0\}$.
2. $W = \{(x, y) \in R^2 | x + y \geq 0\}$.

Example 4.3.6:

Let M be a square matrix of order n,

1. **The set of all upper (lower) triangular matrices is a subspace of M.**
2. **The set of all symmetric matrices is a subspace of M.**

Theorem 4.3.7:

The solution set of the linear homogeneous system in n unknowns is a subspace of F^n (where F is the field of scalars).

Theorem 4.3.8:

Let M and N be two subspaces of the vector space V.

1. **Show that $M \cap N$ is a subspace of V.**
2. **Is $M \cup N$ a subspace of V? If so, prove it. Otherwise explain by giving a counterexample.**

Theorem 4.3.9:

Let W be a subspace of V and M be a subspace of W. Then M is a subspace of V. So the property of subspace is transitive.

Theorem 4.3.10:

Let V be a vector space and $\phi \neq A \subseteq V$. Define the set

$$\text{Span}(A)=\{\vec{v} \mid \vec{v} \text{ is a linear combination of members of } A\}.$$

Then

1. **Span (A) is a subspace of V called a subspace generated by A.**
2. **Span (A) is the smallest subspace of V containing A.**

Example 4.3.8: In R^3, let

$$A = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Find the span(A).

Solution:

$x = \frac{1}{31}(5a-14b+6c)$, $y = \frac{1}{31}(-a+9b+5c)$, $z = \frac{1}{31}(7a-b-4c)$

Example 4.3.9: In R^3, let

$$A = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\}$$

Find the Span (A)

Solution:

Span (*A*)={(*a*,*b*,*c*)|*a*-7*b*+*c*=0}

Example 4.3.10: In R^3, let

$$A = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right\}$$

Find the Span (*A*)

Solution:

$x = \frac{1}{31}(5a-14b+6c+35t)$, $y = \frac{1}{31}(-a+9b+5c-38t)$, $z = \frac{1}{31}(7a-b-4c+18t)$, $u=t$

4.4 Linear dependence and independence

Let *V* be a vector space over a field *F* and $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in *V*. We say that $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if the following implication holds: For any set of scalars $\lambda_1, \ldots, \lambda_n$

$$\sum_{i=1}^{n} \lambda_i \vec{v}_i = 0 \rightarrow \lambda_i = 0$$

Otherwise, $\vec{v}_1, \ldots, \vec{v}_n$ is called linearly dependent.

A subset $L=\{\vec{v}_1, \ldots, \vec{v}_n\}$ is said to be linearly independent if the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent

An infinite set *L* of vectors is linearly dependent or independent according to whether there do or do not exist vectors $\vec{v}_1, \ldots, \vec{v}_n$ in S that are linearly dependent.

Proposition 4.4.1:

Let *V* be a vector space over a field *F* and $L=\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$.

1. **If $\vec{v}_i=0$, for some *i*, then *L* is linearly dependent.**
2. **If $\vec{v}_i = c\vec{v}_k$, for some $1 \le i, k \le n$ and $c \in F$, then *L* is linearly dependent.**
3. **Two vectors $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent if one of them is a multiple of the other.**

4. **If a set *L* of vectors is linearly independent, then any subset of *L* is linearly independent. Alternatively, if *L* contains a linearly dependent subset, then *L* is linearly dependent.**

Example 4.4.2:

In the vector space R^3 over the field *R*, the set

$$A=\left\{\begin{pmatrix}0\\2\\3\end{pmatrix},\begin{pmatrix}1\\-3\\2\end{pmatrix},\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}$$ **is linearly independent.**

Example 4.4.3:

In the vector space R^3 over the field *R*, the set

$$A=\left\{\begin{pmatrix}2\\1\\-1\end{pmatrix},\begin{pmatrix}0\\3\\-2\end{pmatrix},\begin{pmatrix}1\\4\\-2\end{pmatrix}\right\}$$ **is linearly independent.**

Example 4.4.4:

In the vector space R^3 over the field *R*, the set

$$A=\left\{\begin{pmatrix}-4\\2\\3\end{pmatrix},\begin{pmatrix}1\\-2\\-1\end{pmatrix},\begin{pmatrix}-2\\-2\\1\end{pmatrix}\right\}$$ **is linearly dependent.**

Solution:

$x=-t$, $y=-2t$, $z=t$.

Theorem 4.4.5:

Let *V* be a vector space over e field *F*. The vectors $\vec{v}_1,\dots,\vec{v}_k$ in *V* are linearly dependent if and only if some $\vec{v}_i$ is a linear combination of the others.

Theorem 4.4.6:

In a vector space *V* over a filed *F*, if *A*=$\{\vec{v}_1,\dots,\vec{v}_n\}$ is a linearly independent set, then any vectors has a unique linear combination representation.

Fundamental result in linear algebra

Theorem 4.4.7:

In a vector space *V* over a field *F*, if *S*={$\vec{v}_1$,…, $\vec{v}_s$} is a span of *V* and *A*={$\vec{w}_1$,…, $\vec{w}_t$} is a linearly independent set, then *t*≤*s*.

Proof:

For the set *A*, consider

$$\sum_{j=1}^{t} x_j \vec{w}_j = 0 \dots (1)$$

Since *S* is a span of *V*, then

$$\vec{w}_j = \sum_{i=1}^{s} a_{ij} \vec{v}_i \dots (2)$$

From (1) and (2), we obtain

$$\sum_{j=1}^{t} x_j \sum_{i=1}^{s} a_{ij} \vec{v}_i = 0$$

$$\sum_{i=1}^{s} \left(\sum_{j=1}^{t} a_{ij} x_j \right) \vec{v}_i = 0$$

Then, we obtain

$$\sum_{j=1}^{t} a_{ij} x_j = 0, \text{for } i = 1, \dots, s$$

In other words, we obtain

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1t}x_t = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2t}x_t = 0 \\ \vdots \\ a_{s1}x_1 + a_{s2}x_2 + \cdots + a_{st}x_t = 0 \end{gathered}$$

The matrix homogeneous system of the above system is written as follows

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1t} \\ a_{21} & a_{22} & \cdots & a_{2t} \\ \cdot & \cdot & \cdot & \cdot \\ a_{s1} & a_{s1} & \cdots & a_{st} \end{pmatrix}_{s\times t} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This matrix coefficient of the homogeneous system is of order $s\times t$.

According to Corollary 3.5.2:

If $t>s$, then there is a nontrivial solution. Hence $x_k \neq 0$, for some $1\leq k\leq t$, consequently A is L.D (Impossible).

As a result, we obtain $t\leq s$.

Theorem 4.4.8:

The nonzero rows of a matrix in echelon form are linearly independent.

Example 4.4.9:

Consider the following matrix

$$A=\begin{pmatrix} 2 & 1 & 3 & 5 & 0 & 1 \\ 0 & 0 & 3 & 7 & 5 & 6 \\ 0 & 0 & 0 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly the nonzero rows are r_1, r_2, r_3 and r_4. Let

λ_1 (2, 1, 3, 5, 0, 1)+ λ_2 (0, 0, 3, 7, 5, 6)+ λ_3 (0, 0, 0, 4, 1, 3)+ λ_4 (0, 0, 0, 0, 5, 3)= (0, 0, 0, 0, 0, 0)

($2\lambda_1$, λ_1, $3\lambda_1$, $5\lambda_1$, 0, λ_1)+ (0, 0, $3\lambda_2$, $7\lambda_2$, $5\lambda_2$, $6\lambda_2$)+ (0, 0, 0, $4\lambda_3$, λ_3, $3\lambda_3$)+ (0, 0, 0, 0, $5\lambda_4$, $3\lambda_4$)= (0, 0, 0, 0, 0, 0)

Then, we obtain

($2\lambda_1$, λ_1, $3\lambda_1+3\lambda_2$, $5\lambda_1+7\lambda_2+4\lambda_3$, $5\lambda_2+\lambda_3+5\lambda_4$, $\lambda_1+6\lambda_2+3\lambda_3+3\lambda_4$)= (0, 0, 0, 0, 0, 0)

Then

$$\left.\begin{array}{l} 2\lambda_1=0 \\ \lambda_1=0 \\ 3\lambda_1+3\lambda_2=0 \\ 5\lambda_1+7\lambda_2+4\lambda_3=0 \\ 5\lambda_2+\lambda_3+5\lambda_4=0 \\ \lambda_1+6\lambda_2+3\lambda_3+3\lambda_4=0 \end{array}\right\} \dots \text{(I)}$$

Clearly, the solution of System (I) are $\lambda_1=0$, $\lambda_2=0$, $\lambda_3=0$ and $\lambda_4=0$.

Then r_1, r_2, r_3 and r_4 are L.I.

4.5 Basis and dimension

Definition 4.5.1:

Let V be a vector space over a field F. A subset $\beta=\{\vec{v}_1, \dots, \vec{v}_n\}$ is called a basis of V if

1. **β spans V.**
2. **β is linearly independent.**

Example 4.5.2:

For the vector space R^3 over the filed R, the set

$\beta=\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of R^3 called the standard basis.

Generally, for R^n, the set

$\beta=\{(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,1)\}$ is the standard basis of R^n.

Example 4.5.3:

Show that the set $\beta=\{(1,-1,2), (0,1,2), (3,-2,1)\}$ is a basis of the vector space R^3 over the field R.

Definition 4.5.4:

Let V be a vector space over a field F for which β is a basis. If $|\beta|=n$ (the cardinality of β), then we say that V has a finite dimension or n-dimension. We write

***dim* $(V)=n$.**

The vector space {0} has 0-dimension.

In case, if a vector space has not a finite basis, then V is called an infinite-dimensional vector space.

Theorem 4.5.5:

Let β_1 and β_2 be two bases of the vector space V over a field F for which $|\beta_1|=m$ and $|\beta_2|=n$, then $m=n$.

Proof:

From Fundamental result in linear algebra, we obtain $m\leq n$ and $n\leq m$, then $m=n$.

Example 4.4.6:

For the vector space P_3 over a field R, the set $\{1, x, x^2, x^3\}$ is a basis of P_3. Then *dim* $(P_3)=4$. Generally *dim* $(P_n)=n+1$.

Solution:

$p\in P_3 \rightarrow p=ax^3+bx^2+cx+d$, then $p=a(x^3)+b(x^2)+c(x)+d(1)$

Example 4.4.7:

Let V be the set of all symmetric matrices of order two. Find *dim* (V) (Explain your answer).

Solution:

$V=\{A_{2\times2}|A=A^T\}$.

$A\in V \rightarrow A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

Set the following set

$\beta=\left\{\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\right\}$

For $A\in V$, we may select $\lambda_1=a$, $\lambda_2=b$ and $\lambda_3=c$.

Clearly,

$$A=a\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}+ b\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}+ c\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Hence β spans V.

For any scalars λ_1, λ_2 and λ_3 such that:

$$\lambda_1\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}+ \lambda_2\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}+ \lambda_3\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}=\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$\rightarrow\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix}=\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Then, we obtain $\lambda_1=0$, $\lambda_2=0$ and $\lambda_3=0$.

Consequently β is linearly independent.

As a result β is a basis of V.

Exercise 4.4.8:

Let V be the set of all symmetric matrices of order three. Find *dim* (V) (Explain your answer).