Question Bank

Introduction to Algebraic Geometry

2022-2023

Q1: Every ascending chain of ideals in $F[x_1,\ldots,x_n]$ is stabilised.

Q2: In an affine space $\mathbb{A}^n$, any affine variety can be written as a finite union of irreducible affine varieties.

Q3: Let $J=<f_1,\ldots,f_m>$ be an ideal in $F[x_1,\ldots,x_n]$. Prove that $V(J)=V(f_1,\ldots,f_m)$.

Q4: Define and give an example for each of the following,

1. Affine subspace. 2. Algebraically closed field. 3. Affine variety. 4. Algebraic set. 5. Zariski closure of a set. 6. Vanishing ideal $I(B)$ of B. 7. Irreducible ideal.

Q5: State and prove The Strong Nullstellensatz.

Q6: Let F be an arbitrary field,

i. For any affine varieties V_1 and V_2 of $\mathbb{A}^n$, prove that

$$V_1\subseteq V_2 \leftrightarrow I(V_2)\subseteq I(V_1)$$

ii. For any ideals J_1 and J_2 of $F[x_1,\ldots.x_n]$, prove that

$$J_1\subseteq J_2 \rightarrow V(J_2)\subseteq V(J_1)$$

Q7: Let I and J be two ideals in $F[x_1,\ldots,x_n]$,

1. Define $I+J$.
2. Prove that $I+J$ is an ideal of $F[x_1,\ldots,x_n]$.
3. Prove that $I+J$ is the smallest ideal containing I and J.
4. $I=<f_1,\ldots,f_r> \wedge J=<g_1,\ldots,g_s> \rightarrow I+J=<f_1,\ldots,f_r,g_1,\ldots,g_s>$.

Q8: Let I and J be ideals in $F[x_1,\ldots,x_n]$, prove that

$$\sqrt{I\cap J}=\sqrt{I}\cap\sqrt{J}$$

Q9: Let F be an infinite field and $f,g\in F[x_1,\ldots,x_n]$. Then f and g are equal polynomials if and only if f and g are the same functions.

Q10: The union of two algebraic sets is an algebraic set.

Q11: Let $f_1,\ldots, f_s$ be polynomials in $K[x_1,\ldots, x_n]$, then

$$<f_1,\ldots, f_s> \subseteq I(V(<f_1,\ldots, f_s>)).$$

Q12: State and prove Hilbert's Nullstellensatz

Q13: Let V be an affine variety of $\mathbb{A}^n$. If $f^m \in I(V)$, for some +ve integer m, then $f \in I(V)$.

Q14: Let F be an algebraically closed field and M be an ideal of $F[x_1,\ldots,x_n]$. Then

$$\sqrt{M} = I\ (V\ (M))$$

Q15: Let $S=\{(-3,-1),(-4,5),(2,-4)\}$. Find $I(S)$.

Q16: Prove or disprove:

Let J be an ideal of $F[x_1,\ldots,x_n]$, then $V(J)=V(I(V(J)))$

Q17: Let X and Y be two algebraic subsets of an affine space such that $X\subseteq Y$, then $Y=X\cup(\overline{Y-X})$.

Q18: For any subset B of $\mathbb{A}^n$,

$$\bar{B} = V\big(I(B)\big).$$

Q19: Let $f_1,\ldots,f_r \in F[x_1,\ldots,x_n]$, then $<f_1,\ldots,f_r>=<f_1>+\ldots+<f_r>$.

Q20: If I and J are ideals in $F[x_1,\ldots,x_n]$, then $V(I+J)=V(I)\cap V(J)$.

Q21: If I and J are ideals in $F[x_1,\ldots,x_n]$, then $V(I.\ J)=V(I)\cup V(J)$.

Q22: Sketch the following affine varieties,

1. $V(x^2-y^2)$ in $\Re^2$.
2. $V(2x+y-1,3x-y+2)$ in $\Re^2$.
3. $V(xz^2-xy)$ in $\Re^3$.
4. $V(x+2y, -x-y)$ in $\Re^3$.

Q23: Let F be an infinite field and $f\in F[x_1,\ldots, x_n]$. Then f is the zero polynomial in $F[x_1,\ldots, x_n]$ if and only if f is the zero function.

Q24: Prove that every vector space is an affine space.

Q25: Let $\{I_\alpha | \alpha \in \Delta\}$ be a set of ideals in $F[x_1, \ldots, x_n]$, where Δ is arbitrary. Prove that

$$V\left(\bigcup_{\alpha\in\Delta} I_\alpha\right) = \bigcap_{\alpha\in\Delta} V(I_\alpha)$$

Q26: What is Zariski topology on an affine space $\mathbb{A}^n$? Prove that Zariski topology satisfies the conditions of a topology on $\mathbb{A}^n$.

Q27: Let $f_i, g_j \in F[x_1, \ldots, x_n]$, where $1 \le i \le r$ and $1 \le j \le s$ (generally $r \ne s$). Then

1. $V(f_1, \ldots, f_r) \cup V(g_1, \ldots, g_s) = V(\{f_i g_j | 1 \le i \le r \text{ and } 1 \le j \le s\})$.
2. $V(f_1, \ldots, f_r) \cap V(g_1, \ldots, g_s) = V(f_1, \ldots, f_r, g_1, \ldots, g_s)$.