Mathematics Department | Year Four | General Topology

Semester I | Final Exam | First Trial (Answer)

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Q1:

1. F
2. T
3. T
4. F
5. F
6. F
7. F
8. T
9. F
10. F
11. F
12. T
13. F
14. F
15. T

Q2:

1. $\{Z^+,\{2,3,4,\ldots\},\{3,4,5,\ldots\},\{4,5,6,\ldots\},\{5,6,7,\ldots\},\{6,7,8,\ldots\}, \{7,8,9,\ldots\}\}$.
2. $\{\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6,7,8\},\{1,2,3,4,5,6,7,8,9\},$
   $\{1,2,3,4,5,6,7,8,9,10\},\ldots,Z^+\}$.
3. $\{14, 15, 16, \ldots\}$.
4. $\{1,2,3,\ldots,103\}$.
5. $\tau_Y=\{\{8,10,11,12,\ldots\}\}\cup\{\{m+9, m+10, m+11,\ldots\};m\in Z^+\}\cup\{Y\}$.

Q3:

(a) Let $\mathfrak{B}$ be the base for τ generated by δ and $\mathfrak{B}_Y$ be the corresponding base for τ_Y generated by $\mathfrak{B}$.

Let $B\cap Y\in\mathfrak{B}_Y$,

$$B \cap Y \in \mathfrak{B}_Y \to B \in \mathfrak{B} \to B = \bigcap_{i=1}^{n} s_i, \text{for some } s_i \in \delta \to B \cap Y = \left(\bigcap_{i=1}^{n} s_i\right) \cap Y$$

$$= \left(\bigcap_{i=1}^{n} (s_i \cap Y)\right)$$

(b)

We have to show that,

$$(\tau_Y)_Z = \tau_Z$$

Let $K \in (\tau_Y)_Z$

$K \in (\tau_Y)_Z \to K = Z \cap H$, for some $H \in \tau_Y$

$H \in \tau_Y \to H = Y \cap G$, for some $G \in \tau$

Then,

$K = Z \cap H \to K = Z \cap (Y \cap G) \to K = (Z \cap Y) \cap G \to K = Z \cap G$.

So, $K = Z \cap G$, for some $G \in \tau$, then $(Z, (\tau_Y)_Z)$ is a subspace of (X, τ).

Let $K \in \tau_Z$

$K \in \tau_Z \to K = Z \cap G$, for some $G \in \tau$

On the other hand,

$G \in \tau \to Y \cap G \in \tau_Y \to Z \cap (Y \cap G) \in (\tau_Y)_Z$

$\to (Z \cap Y) \cap G \in (\tau_Y)_Z \to Z \cap G \in (\tau_Y)_Z \to K \in (\tau_Y)_Z$

Q4:

(a) Disproof

Let $X=\{a, b, c\}$, $\tau_1=\{\phi, \{a\}, X\}$ and $\tau_1=\{\phi, \{b\}, X\}$ be two topologies on X.

$\tau_1 \cup \tau_2 = \{\phi, \{a\}, \{b\}, X\}$.

Clearly, $\tau_1 \cup \tau_2$ is not a topology on X, since $\{a\} \cup \{b\} = \{a, b\} \notin \tau_1 \cup \tau_2$.

(b) Proof

Let A be an open set in X,

A be an open set in $X \to A^c$ is a closed set in $X \to clo(A^c) = A^c$.

Then,

$$bou(A)\cap A=\big(clo(A)\cap clo(A^c)\big)\cap A$$

$$=\big(clo(A)\cap A^c\big)\cap A$$

$$=clo(A)\cap\big(A^c\cap A\big)=clo(A)\cap\phi=\phi.$$

Conversely

Let $bou(A)\cap A=\phi$.

Then

$$\big(clo(A)\cap clo(A^c)\big)\cap A=\phi\rightarrow\big(clo(A)\cap A\big)\cap clo(A^c)=\phi$$

$$\rightarrow A\cap clo(A^c)=\phi,\text{ since } A\subseteq clo(A).$$

$$\rightarrow A\subseteq\big(clo(A^c)\big)^c$$

$$\rightarrow A\subseteq int(A).$$

Since $int(A)\subseteq A$, then $A=int(A)$, hence $A\in\tau$.

(c) Proof

Let $y\in ext_Y(A)$,

$$y\in ext_Y(A)\rightarrow y\in int_Y(Y\text{-}A)\rightarrow y\in H\subseteq Y\text{-}A,\text{ for some } H\in\tau_Y$$

$$\rightarrow y\in G\cap Y\subseteq Y\text{-}A,\text{ for some } G\in\tau.$$

Clearly,

$G\cap A=\phi\rightarrow G\subseteq X\text{-}A$.

Hence,

$G\subseteq X\text{-}A\rightarrow y\in int(X\text{-}A)\rightarrow y\in ext(A)\rightarrow y\in Y\cap ext(A)$.

Let $y\in Y\cap ext(A)$,

$$y\in Y\cap ext(A)\rightarrow y\in Y\wedge y\in ext(A)\rightarrow y\in Y\wedge y\in int(X\text{-}A)$$

$$\rightarrow y\in Y\wedge y\in G\subseteq X\text{-}A$$

Clearly,

$G\subseteq X\text{-}A\rightarrow G\cap Y\subseteq Y\text{-}A$.

Since $y\in G\cap Y$, and $G\cap Y$ is open in Y, then $y\in int_Y(Y\text{-}A)=ext_Y(A)$