Mathematics Department Year Four General Topology

Semester II Final Exam First Trial (Answer)

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Q1:

1. T
2. T
3. F
4. F
5. T
6. F
7. T
8. T
9. F
10. T
11. F
12. F
13. T
14. T
15. F

Q2:

a. Clearly, F and G^c are disjoint closed sets in X

$\rightarrow \exists\ H, H_1 \in \tau$ such that $F \subseteq H$, $G^c \subseteq H_1$ and $H \cap H_1 = \phi$

$\cong \exists\ H, H_1 \in \tau$ such that $F \subseteq H$, $G^c \subseteq H_1$ and $H \subseteq H_1^c$

$\cong \exists\ H, H_1 \in \tau$ such that $F \subseteq H$, $H_1^c \subseteq G$ and $H \subseteq H_1^c$

$\cong \exists\ H \in \tau$ such that $F \subseteq H$ and $clo(H) \subseteq clo(H_1^c) = H_1^c \subseteq G$

b.

Let (X, τ) be not compact, then there exists a class of open sets $\{G_\alpha\}$ such that,

$$X = \bigcup_{\alpha\in\Delta} G_\alpha \text{, while } X \neq \bigcup_{\alpha\in\Delta^*} G_\alpha \text{ for any finite subset } \Delta^* \text{ of } \Delta.$$

Then $\{G_\alpha{}^c\}$ is a class of closed subsets of X for which

$$\bigcap_{\alpha\in\Delta^*} G_\alpha{}^c \neq \phi, \text{for any finite subsets } \Delta^* \text{ of } \Delta \rightarrow \bigcap_{\alpha\in\Delta} G_\alpha{}^c = \phi$$

c. Let $f\,(X,\tau)\rightarrow(Y,\rho)$ be continuous and $B\subseteq Y$.

$\rightarrow f^{-1}(int(B))\in\tau$

Since $f^{-1}(int(B))\subseteq f^{-1}(B)$ and $int(f^{-1}(B))$ is the largest open set contained in $f^{-1}(B)$, then $f^{-1}(int(B))\subseteq int(f^{-1}(B))$

Conversely, let $G\in\rho$, then by hypothesis, $f^{-1}(int(G))\subseteq int(f^{-1}(G))$

$\rightarrow f^{-1}(int(G))= f^{-1}(G)\subseteq int(f^{-1}(G))\rightarrow\subseteq f^{-1}(G)\in\tau$

Q3:

(a)

1. $\tau_1=\{\phi,\{a\},\{a, b\},X\}$, $\tau_2=\{\phi,\{a\},\{b\},\{a, b\},X\}$ and $\tau_3=\{\phi,\{a\},\{b\},\{a, b\}, \{a, b, c\}, X\}$
2. $\tau_1=\{\phi,\{a, b\},\{c, d\},X\}$, $\tau_2=\{\phi,\{a\}, \{b, c\}, \{a, b, c\},\{b, c, d\},X\}$ and $\tau_3=\{\phi,\{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\},\{b, c, d\}, X\}$

(b) Let $A\subseteq X$, then by hypothesis, $f\,(clo(A))$ is closed in Y.

On the other hand, $f\,(A)\subseteq f\,(clo(A))$, then $clo(\,f\,(A))\subseteq f\,(clo(A))$

Q4:

(a) proof

To show that (X,τ) is T_1, let $x, y\in X$ and $x\neq y$

(X,τ) is $T_0\rightarrow clo(\{x\})\neq clo(\{y\})$

Let $z\in clo(\{x\})$- $clo(\{y\})$

$z \notin clo(\{y\})$ and $clo(\{y\})$ is a closed set$\rightarrow \exists G_z$, $H \in \tau$ such that $z \in G_z$, $clo(\{y\}) \subseteq H$ and $G_z \cap H = \phi$. Notice that $y \in H$.

From $z \in clo(\{x\}) \rightarrow x \in G_z$ and $x \notin H$.

On the other hand,

$z \in clo(\{x\})$- $clo(\{y\}) \rightarrow z \notin clo(\{y\}) \rightarrow \exists K_z \in \tau$ such that $y \notin K_z$

For K_z, we have $x \in K_z$, since $z \in clo(\{x\})$.

Hence, for the open sets H, K_z, we have

$x \in K_z$, $y \in H$, $y \notin K_z$ and $x \notin H$.

Similarly, if $z \in clo(\{y\})$-$clo(\{x\})$

(b) Disproof

Let $X=\{a, b, c\}$, $Y=\{0, 1, 2\}$, $\tau=\{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\rho=\{\phi, \{1\}, \{1, 3\}, Y\}$. Define $f:(X,\tau) \rightarrow (Y,\rho)$ as follows,

$f(a)=f(b)=1$ and $f(c)=3$

Let $A=\{b, c\}$, then $\tau_A=\{\phi, \{b\}, \{c\}, Y\}$.

To show that f is an open map, we have,

$f(\phi)=\phi$, $f(\{a\})=\{1\}$, $f(\{a, b\})=\{1\}$, $f(\{a, c\})=\{1, 3\}$ and $f(X)=\{1,3\}$.

While, the restriction map $f|_A:(A,\tau_A) \rightarrow (Y,\rho)$ is not an open map, since

$f|_A(\{c\})=\{3\} \notin \rho$.