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New Solution for Harmonic Oscillator and LMG and Double Beta Decay models Hamiltonian based on the Recurrence Formula and Jacobi matrix methods

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Abstract

Harmonic oscillator is one of the most important and beautiful models in physics.There are three various methods,Analitical and algebraic and Approximation ,for solving Quantum harmonic oscilatore. Quantum harmonic oscilatore is used to introduce the Recurrence Formula and Jacobi matrix concept easily.A Numeric Solution for Harmonic Oscillator and lipkin-meshkov-glick(LMG)and double Beta decay models Hamiltonian(that are based on pairing concept ,the part of the fermion Hamiltonian) are obtained by using the Recurrence Formula and Jacobi matrix.we compared analytical method with numeric method for a harmonic oscillator. we have studied the shape- phase transitions for one example of nuclei. At the end we compare the results thus obtained with those of Beth- ansatz and Hartree- Fock methods. Many physical systems (nuclei, molecules, atomic clusters, etc.) are characterized in their equilibrium configuration by a shape. These shapes are in many cases rigid. However, there are several situations in which the system is rather floppy and undergoes a phase transition between two different shapes. There isn't any restriction on the parameters specifying the strength of the interactions.

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1 Introduction

Physicists always use Harmonic oscillator for initial concepts.It has important role in physics.It is due to its usefullness to describe the dynamics of many physical systemes.It can model the bond in a molecule as a spring connecting two atoms and use the harmonic oscillator expression to describe the potential energy for the periodic vibration of the atoms.The isotropic three-dimensional harmonic oscillator potential allow analytical solutions.As we know the nuclear many body problems are often aproximated by mean filed or liquid drop models for study the shape and the spectrum of low lying single particle and collective excitations[1-3]. A goal of nuclear physics is to account for the properties of nuclei in terms of mathematical models of their structure and internal motion. Sometimes they combine oscillators for expanded purposes.one of this is pairing interaction.nuclear Pairing are known to play an important role in differnt single-particle and collective aspects of nuclear structure. The pairing interaction is an important part of the fermion Hamiltonian responsible for the Superconducting phase in metals and in nuclear matter or neutron stars.[4]Ultrasmall superconducting grains and atomic nuclei have pairing in their correspondindg finite systems.[5] Also pairing is the significant effect in the nucleus with un fool level that has even proton and neutron .The main feature of the pairing interaction is that it correlates pairs of particles in time-reversed states. Several models proposed for pairing problem such as Richardson model, Double Beta Decay and Lipkin-Meshkov-Glick.The LMG model in its simplest version discribes two shells for nucleons and an intraction between them in diffrent shells.There is tradational testing ground for new approximation techniques,that is numerically solvable.[6,7] And are different standard tools for nuclear structure calculation methods that solve these Hamiltonian similar as Beth-ansatz, QRPA,RPA, Hartree-Fock.[7,8,9] .The structure of the article is the following.In section II, we overview Harmonic oscillator and In section III,The LMG model in section IV and Double Beta Decay models then In section V, Based on the recursion relation formulas and the Jacobi matrix from the theory of orthogonal polynomials, a numeric solution for these Hamiltonian models is given, without considering any restriction on the parameters specifying the strengths of the interactions.In section VI,we conclude and discuss our solutions.

2 Harmonic oscillator

Quantum harmonic oscillator involves square law potential in the Schrodinger equation and is a fundamental problem in quantum mechanics. It can be solved by various conventional methods such as (i) analytical methods where Hermite polynomials are involved, (ii) algebraic methods where ladder operators are involved, and (iii) approximation methods where perturbation, variational, semiclassical, etc. techniques are involved.For investigation of Recursion Formula and Jacobi Matrix we will consider simple Harmonic oscillator. At first We will solve it analytically then compare with this method.cosider a vertical spring with hardship coefficient K that a body with mass m hanging it(as figure 1) . Total force obtained when weight(mg) added to it and Hamiltonian written as below:

$$H = \frac{p^2}{2m} + \frac{1}{2}mw^2x^2 - mgx \tag{2-1}$$

Consider X_0 as movement from new equilibrium that obtained when F=0

$$mw^2x - mg = 0 \tag{2-2}$$

$$X_0 = \frac{g}{w^2} \tag{2-3}$$

Then we will define new coefficient as

$$X = x - \frac{g}{w^2} \tag{2-4}$$

Hamiltonian written with new confidents is

$$H = \frac{p^2}{2m} + \frac{1}{2}mw^2(x - \frac{g}{w^2})^2 - \frac{1}{2}mw^2(\frac{g^2}{w^4}) \tag{2-5}$$

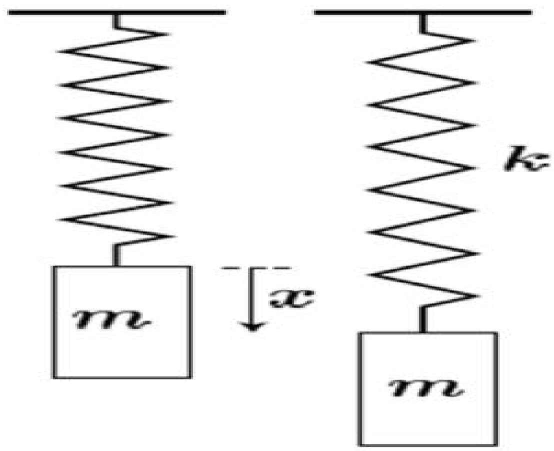

Figure 1: Harmonic Oscilator

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Then new Energy spectrum by old spectrum is written as:

$$E_f = E_i - \frac{1}{2} m \frac{g^2}{w^2} \tag{2-6}$$

The Hamiltonian of the Harmonic oscillator model can be written as:

$$H = \hbar w(a^\dagger a + \frac{1}{2}) - mg\sqrt{\frac{\hbar}{2mw}}(a + a^\dagger) \tag{2-7}$$

We will choice unit vector as

$$|\psi_n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}} \tag{2-8}$$

After effecting Hamiltonian on it we have

$$a^\dagger a|\psi_n\rangle = a^\dagger a \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle = a^\dagger (aa^\dagger)\frac{a^{(n-1)}}{\sqrt{n!}}|0\rangle = \qquad (2\text{-}9)$$

$$a^\dagger(a^\dagger a + 1)\frac{a^{\dagger(n-1)}}{\sqrt{n!}}|0\rangle = \frac{a^{\dagger 2}(aa^{\dagger(n-1)})}{\sqrt{n!}}|0\rangle + \frac{a^{\dagger(n)}}{\sqrt{n!}}|0\rangle = |\psi_n\rangle + \frac{a^{\dagger 2}aa^\dagger(a^{\dagger(n-2)})}{\sqrt{n!}}|0\rangle = \qquad (2\text{-}10)$$

$$|\psi_n\rangle + a^{\dagger 2}(a^\dagger a + 1)\frac{a^{\dagger(n-2)}}{\sqrt{n!}}|0\rangle = 2|\psi_n\rangle + \frac{a^{\dagger 3}aa^\dagger(a^{\dagger(n-2)})}{\sqrt{n!}}(2\text{-}11)$$

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$$= n|\psi_n\rangle \tag{2-12}$$

the first hamiltonian section written by numeric operator will be:

$$\hbar w(a^\dagger a + \frac{1}{2}) = \hbar w(n + \frac{1}{2}) \tag{2-13}$$

considering the other sentences and effecting $|\psi_n\rangle$ on them

$$a|\psi_n\rangle = a\frac{a^{\dagger(n)}}{\sqrt{n!}}|0\rangle = (a^\dagger a + 1)\frac{a^{\dagger(n-1)}}{\sqrt{n!}}|0\rangle = (a^\dagger a + 1)\frac{a^{\dagger(n-1)}}{\sqrt{n!}}|0\rangle = a^\dagger a\frac{a^{\dagger(n-1)}}{\sqrt{n!}}|0\rangle + \frac{a^{\dagger(n-1)}}{\sqrt{n(n-1)!}}|0\rangle \tag{2-14}$$

$$a^\dagger(a^\dagger a + 1)\frac{a^{\dagger(n-2)}}{\sqrt{n!}}|0\rangle + \frac{1}{\sqrt{n}}|\psi_{(n-1)}\rangle = \frac{a^{\dagger 2}a(a^{\dagger(n-2)})}{\sqrt{n!}}|0\rangle + \frac{2}{\sqrt{n}}|\psi_{(n-1)}\rangle \tag{2-15}$$

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$$= \frac{n}{\sqrt{n}}|\psi_{(n-1)}\rangle = \sqrt{n}|\psi_{(n-1)}\rangle \tag{2-16}$$

$$a^\dagger|\psi_{(n)}\rangle = a^\dagger\frac{a^{n\dagger}}{\sqrt{n!}}|0\rangle = \frac{(a^{\dagger(n+1)})}{\sqrt{n!}}|0\rangle = \sqrt{n+1}|\psi_{(n+1)}\rangle \tag{2-17}$$

Then from collecting the above results we will have:

$$H|\psi_{(n)}\rangle = \hbar w(n + \frac{1}{2})|\psi_{(n)}\rangle \tag{2-18}$$

and for the secound term we have

$$-mg\sqrt{\frac{n\hbar}{2mw}}|\psi_{n-1}\rangle - mg\sqrt{\frac{(n+1)\hbar}{2mw}} \tag{2-19}$$

from comparing the upper relations with recurrence formula in section V, we will distinguish the α_n and β_n as:

$$\alpha_n = \hbar w(n + \frac{1}{2}), \quad \beta_n = mg\sqrt{\frac{n\hbar}{2mw}}, \tag{2-20}$$

With supposing coefficients as $w = 10$, $m = 1$, $\hbar = 1$ so we will have:

$$\alpha_2 = \hbar w(2 + \frac{1}{2}) = 25 \qquad \alpha_1 = \frac{3}{2}\hbar w = 15 \tag{2-21}$$

$$\beta_1 = \sqrt{1} * 1 * g * \sqrt{\frac{1}{2 * 1 * 10}} = \frac{g}{\sqrt{20}} = 0.223g \tag{2-22}$$

$$\beta_2 = \sqrt{2} * 1 * g * \sqrt{\frac{1}{2 * 1 * 10}} = g * \frac{2}{\sqrt{20}} = 0.223g = 0.316g \tag{2-23}$$

For example the first three recursion sentencec will be:

$$q_0 = 1 \qquad q_1 = x - \alpha_1 = x - 15 \tag{2-24}$$

$$q_2 = (x - \alpha_2)q_1 = (x - 25)(x - 15) + 0.223g^2 \tag{2-25}$$

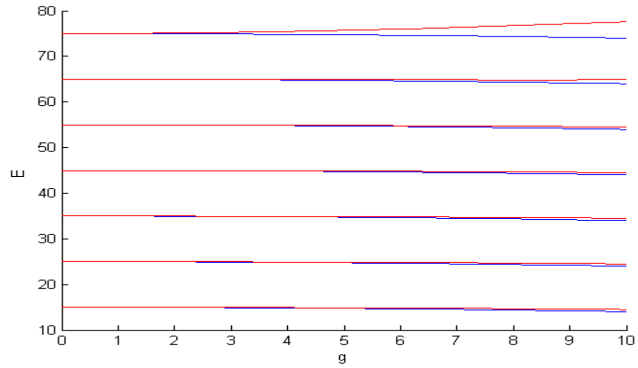

Figure 2: comparing analytical method(blue colour) with numerical method (red colour)

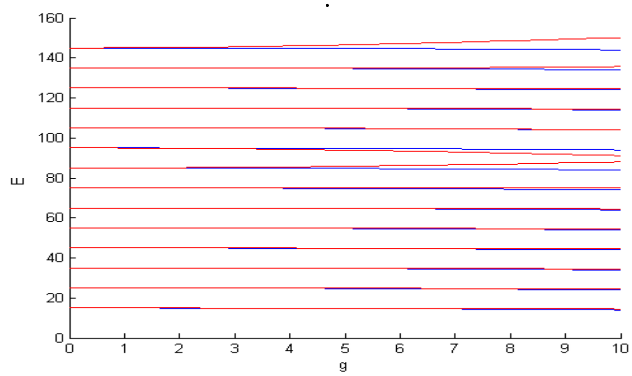

Figure 3: comparing analytical method(blue colour) with numerical method (red colour)

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graphicx

3 Lipkin-Meshkov-Glick (LMG) model

This model is offered in 1965 for explaining phase transformation in nucleus. Then is used for quantum information and quantum phase transformation. In this model,N particles can distribute themselves on two N-fold degenerate levels distinguished by a quantum number σ with $= \pm 1$ .Let $a^{\dagger}_{p\sigma}$ be fermion creation -annihilation operator for a particle in the P state and level with P=1,2,. . . .,N. The Hamiltonian of the LMG model can be written as [5,6]

$$H = \varepsilon \sum_{p,p'=1,2,..N;\sigma=\pm1} \sigma a^{\dagger}_{p\sigma}; a_{p\sigma} + V \sum_{p,p'=1,2,..N;\sigma=\pm1} a^{\dagger}_{p\sigma} a^{\dagger}_{p'\sigma} a_{p'-\sigma} a_{p-\sigma} + W \sum_{p,p'=1,2,..N;\sigma=\pm1} a^{\dagger}_{p\sigma} a^{\dagger}_{p'-\sigma} a_{p'\sigma} a_{p-\sigma} \tag{3-26}$$

Where V and W are parameters specifying the strengths of the interactions.The interaction term V scatters a pair of particles across the Fermi level, it is a two particle hole interaction. The term proportional to W scatters one particle up and another down. By introducing the so called pseudo spin operators

$$J_+ = \sum_p a^{\dagger}_{p+} a_{p-}, \quad J_- = \sum_p a^{\dagger}_{p-} a_{p+}, \quad J_z = 1/2 \sum_{p\sigma} \sigma a^{\dagger}_{p\sigma} a_{p\sigma} \tag{3-27}$$

That follows from the following relations

$$[J_+, J_-] = J_z, \quad [J_z, J_\pm] = \pm J_\pm \tag{3-28}$$

In the SU(2)representation Hamiltonian be written as

$$H = \epsilon J_z + \frac{1}{2} V (J_+^2 + J_-^2) + \frac{1}{2} W (J_+ J_- + J_- J_+) \tag{3-29}$$

4 Double Beta Decay

This decay is one of the weak rare nuclear processes and happens in the more than sixty isotopes in the periodic table. But experimentally happens only between ten of them including ;

T^{128}, Cd^{116},Mo^{100},Zr^{96},Se^{82},Ge^{76},Ca^{48},Ca^{238},Nd^{150},Te^{130}

this decay can happen in three faces. 1-$\beta\beta 2\upsilon$: Neutrino less double beta decay is one of the best probes for physics beyond the standard model of electroweak interactions. Its existence is tied to fundamental aspects of particle physics such as lepton number no conservation, neutrino mass, right-handed currents in the electroweak interaction, a massless Goldstone boson Majoron, the structure of the Higgs' sector and super symmetry. 2-$\beta\beta 0\upsilon$: Double Beta decay is transition between a nucleus of charge Z and mass number A(with both A and Z even) and one of the same mass and charge Z+2 . 3-$\beta\beta 0 \upsilon\chi$: A model many-body Hamiltonian describing a heterogonous system of Paired protons and paired neutrons and interacting among themselves through Monopole particle-hole and monopole particle-particle interactions is used to Study the double beta decay of Fermi type.The model Hamiltonian is transformed in a many body quasiparticle operator which

$H =\epsilon(\mathrm{N}_p + N_n)+\lambda_1 a^\dagger a+\lambda_2(a^\dagger a^\dagger + aa)$ (4-30)

$Where$ N_P and N_n are proton and neutron quasi particle number operators,respectively,while A, $A^\dagger$ stand for proton neutron pairing operators built up with the quasi particle operators

$$A^\dagger = [a_p^\dagger a_n^\dagger]_{00}, \qquad A = (A^\dagger)^\dagger \tag{4-31}$$

This Hamiltonian is the Hamiltonian that is set mixture of proton and neutron sphere shell model and obtain from particle-particle and particle-hole proton-neutron pairing interaction. If we delete the sentence with coefficient λ_1 it defining the LMG Hamiltonian. [3, 4]

(4-31)

5 Spectrum distribution, Recurrence Formula and Jacobin matrix Formula

We suppose unit vector $|\phi_i\rangle$ with the following representations .

$$|\phi_i\rangle = \begin{cases} |J, J-2k_i\rangle & \text{for } k_i = 0,1,...,J \\ |J, J-(2k_i+1)\rangle & \text{for } k_i = 0,1,...,J-1 \end{cases} . \quad (5\text{-}32)$$

We act the Hamiltonian H on these unit vectors and find the coefficients according to this relation [7]

$$|\phi_i\rangle = \beta_{i+1}|\phi_{i+1}\rangle + \alpha_i|\phi_i\rangle + \beta_i|\phi_{i-1}\rangle \qquad i = 0,1,...,n. \quad (5\text{-}33)$$

Using $2\beta\beta\upsilon$ model for the nucleus with Z=4, j=9/4,$\epsilon = 1mev, N = 6$.coefficients λ_1 and λ_2 will be

$$\lambda_1 = o.52 - 0.96k \qquad \lambda_2 = 0.48(0.5+k)(5\text{-}34)$$

for the even states

$$\alpha_{k_i} = 4k_i +_1 (2k_i - (k_i(2k_i-1))/5), \qquad \beta_{k_i} = \lambda_2(2k_i(2k_i-1)(1-(2k_i-1)/10)(1-(k_i-1)/5)) \quad (5\text{-}35)$$

And for the odd states

$$\alpha_{k_i} = 4k_i +_1 (2k_i - (k_i(2k_i+1))/5) , \qquad \beta_{k_i} = \lambda_2(2k_i(2k_i+1)(1-(2k_i-1)/10)(1-k_i/5)(5\text{-}36)$$

As we have in ref.[7]

$$x\begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ . \\ . \\ . \\ . \\ Q_{n-1}(x) \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 & 0 & 0 & . & . & . & 0 \\ \beta_0 & \alpha_1 & \beta_1 & 0 & . & . & . & 0 \\ 0 & \beta_1 & \alpha_2 & \beta_2 & . & . & . & 0 \\ . & . & . & . & & & & 0 \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ 0 & 0 & 0 & 0 & . & . & . & \alpha_{N-1} \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ . \\ . \\ . \\ . \\ Q_{N-1}(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ \beta_{N-1}Q_N(x) \end{pmatrix} \tag{5-37}$$

If x be a zero of $Q_N(x)$, say $x = x_i$ then we obtain $x_iQ(x_i) = JQ(x_i)$ where J is called Jacobi matrix[7] and the eigenvalues $x_1, ..., x_N$ of that are the zeros of $Q_N(x)$ and the eigenvector corresponding to x_i is $(Q_0(x_i), Q_1(x_i), ..., Q_{N-1}(x_i))^{\dagger}$. If we choose $x = x_i$ then the end matrix will be zero, therefore the coefficients α and β are equal to the coefficients in the recursion relations.

$$Q_0(x) = 1, \qquad Q_1(x) = x - \alpha_1,$$

$$xQ_k(x) = \beta_{k+1}Q_{k+1}(x) + \alpha_{k+1}Q_k(x) + \beta_kQ_{k-1}(x) \quad k \geq 1. \tag{5-38}$$

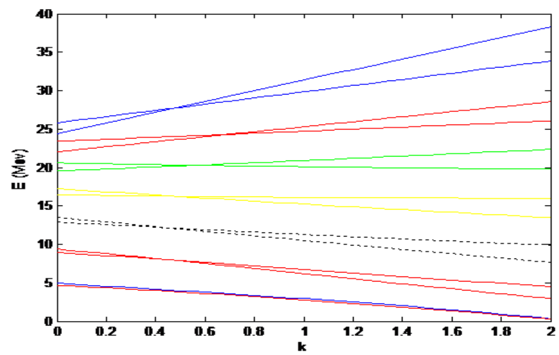

Figure 4: E for Double Beta Decay

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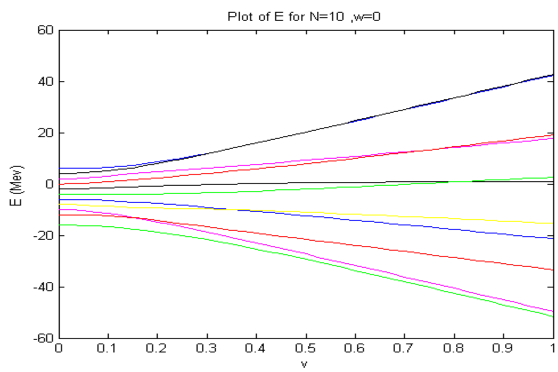

Figure 5: E for N=10 and w=0 in LMG model

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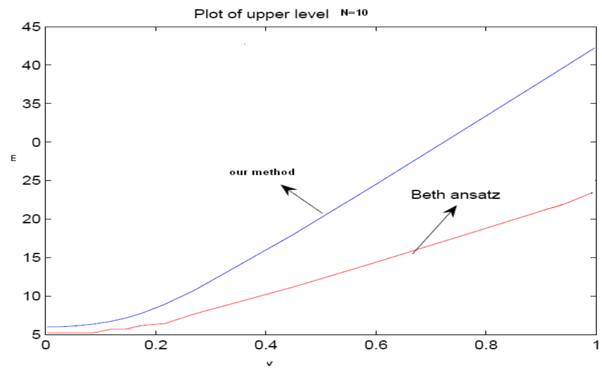
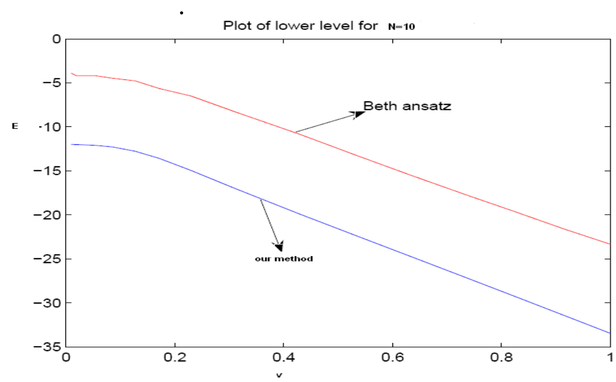
6 Conclusions and Discussions

Many methodes have been used for calculating the energies and wave functions of the Quantume hamiltonian models.In this article we got solutions for the Harmonic Oscillator and Lipkin-Meshkov-Glick and Double Beta Decay models by using the Recurrence Formula and Jacobi matrix methods.In the Harmonic Oscillator graphs,As shown in the figures(1,2,3) we compare analytical methods with numerical methods. clearly shows that thease methods have the same conclutions and their graphs overlap.In the LMG and Double Beta Decay models, Graphs show the pairing transition critical points from normal nucleus to deformed nucleus. There is quantum phase transition in these points.Quantum phase transition shows that the ground state becomes degenerate and a macroscopic change in the ground state energy take place.in this points the ground state and the first excited state come near together and interchange then level crossing take place. There are two types of phase transition in nuclei: the phase transition to super fluidity and to deformation.However, there are several situations in which the system is rather floppy and undergoes a phase transition between two different shapes. A challenging problem is how to describe properties of the system in the phase transition region and in particular at the phase transition point [15]. Shape transition that is shown here has important role in determining their properties such as quadrupole moment, nuclear size and isotope shift. This take place as some control parameters vary along an isotopic (isotonic) chain, for example, brings the system from a spherical to a deformed region. Level-crossing take place in different levels and has seen more in higher level in the stronger interaction regime. Comparing these method with other methods for example Beth-Ansatz ,RPA,QRPA and Hartree-Fock [11,12,13] that have many complicated math formulas we can deduce that spectrum distribution ,recursion relations and Jacobi matrix have more precision and can do desire accounts rapidly. Graphs can be drawn and studied for every N easily.The LMG model was conceived as a test model in nuclear physics. It is simple enough to be solved but it is yet nontrivial. For that reason, since it was established has been used to validate many fermion approximation methods like Hartree- Fock [2]. our high levels is upper than the levels obtain from Bethe ansatz and Hartree- fock methods (Fig.6,7)and the low level is lower (Fig.8), this indicates that ourmethod gives better accuracy with respect to the other methods such as Bethe ansatz [9], and infinite- dimensional algebraic [13] that has sophisticated math, Recurrence Formula and the Jacobi matrix is simple.



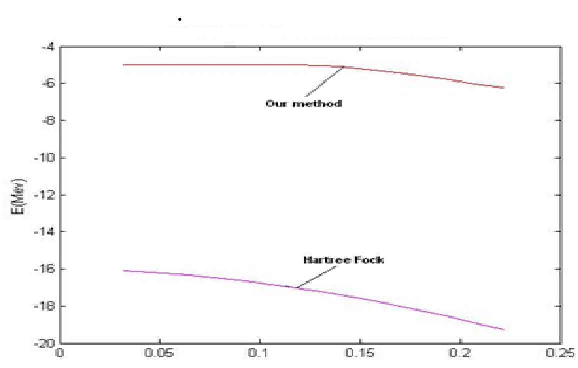
bet ansatz-spec.png

Figure 6: Shows E as a function of the parameter V for 10 particles, W=0,upper level for both Beth ansatz and spectral



bet ansatz-spec.png

Figure 7: Shows E as a function of the parameter V for 10 particles, W=0, Lower level for both Beth ansatz and spectral



Hartree-spec.png

Figure 8: Shows E as a function of the parameter $\chi = \frac{V}{\varepsilon}(\Omega - 1)$ for 10 particles, W=0 Lower level for both Hartree Fock and spectral

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