

Estimation Theory

Department of Statistics & Informative

Fourth Stage

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** Same References as First Semester

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Chapter One

Methods of Estimation

First: Maximum Likelihood Estimation (MLE)

Let X_1, X_2, \dots, X_n be a r.s.s.n from a distⁿ with a p.d.f. $f(x; \theta)$, the joint p.d.f. of X_1, X_2, \dots, X_n denote $L(\theta)$ is called the likelihood function, and the value of $\hat{\theta}$ which maximizes the likelihood function is called Maximum Likelihood Estimator (MLE) for θ , or the m.l.e is solution of:

$$\begin{aligned} j.p.d.f.(x_1, x_2, \dots, x_n) &= f(x_1, x_2, \dots, x_n; \theta) \\ &= L(x_1, x_2, \dots, x_n; \theta) \\ &= \prod_{i=1}^n f(x_i; \theta) \end{aligned}$$
$$\left(\frac{\partial \ln L(\theta)}{\partial \theta} = 0 \quad , \quad \text{with} \quad \frac{\partial^2 \ln L(\theta)}{\partial \theta^2} < 0 \right)$$

Note: If the second derivative less than zero that were the maximum.

The Steps of Maximum Likelihood Estimation

- 1) Find $L(x_1, x_2, \dots, x_n; \theta) = L(\underline{x}; \theta) = \prod_{i=1}^n f(x_i; \theta)$.
- 2) Find $\ln(L(\underline{x}; \theta))$.
- 3) $\frac{\partial \ln(L(\underline{x}; \theta))}{\partial \theta} = 0$.
- 4) Find $\hat{\theta}$.

Ex: Let X_1, X_2, \dots, X_n denote a random sample from Bernoulli distⁿ $\text{Ber}(\theta)$, find the m.l.e for θ .

Sol:

$$\because X \sim \text{Ber}(\theta)$$

$$f(x; \theta) = \theta^x(1 - \theta)^{1-x} \quad , \quad x = 0, 1$$

$\because X$'s are indep.

$$L(\theta) = f(x_1, x_1, \dots, x_1; \theta) = \prod f(x_i; \theta)$$

$$= \theta^{\sum x_i} (1 - \theta)^{n - \sum x_i}$$

$$\ln L(\theta) = \sum x_i \ln(\theta) + (n - \sum x_i) \ln(1 - \theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1 - \theta} \quad , \quad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1 - \theta} = 0$$

$$\frac{(1 - \theta)\sum x_i - \theta(n - \sum x_i)}{\theta^2(1 - \theta)^2} = 0$$

$$\sum x_i - \theta \sum x_i - n\theta + \theta \sum x_i = 0$$

$$\sum x_i - n\theta = 0$$

$$\sum x_i = n\theta \quad \theta_{m.l.e} = \frac{\sum x_i}{n} = \bar{X}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{\sum x_i}{\theta^2} - \frac{n - \sum x_i}{(1 - \theta)^2} < 0$$

• $\theta = \bar{X}$ is m.l.e for θ .

Ex: Let X_1, X_2, \dots, X_n denote a random sample from Poisson distⁿ $\text{Poi}(\theta)$, find the m.l.e for θ .

Sol:

$$\because X \sim \text{Poi}(\theta)$$

$$f(x; \theta) = \frac{e^{-\theta} \theta^x}{x!}, \quad x = 0, 1, \dots$$

• X 's are indep.

$$L(\theta) = f(x_1, x_1, \dots, x_1; \theta) = \prod f(x_i; \theta)$$

$$= \frac{e^{-n\theta} \theta^{\sum x_i}}{\prod (x_i)!}$$

$$\ln L(\theta) = -n\theta + \sum x_i \ln(\theta) - \prod (x_i)!$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -n + \frac{\sum x_i}{\theta}, \quad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$-n + \frac{\sum x_i}{\theta} = 0$$

$$\frac{\sum x_i}{\theta} = n$$

$$\sum x_i = n\theta$$

$$\theta_{m.l.e} = \frac{\sum x_i}{n} = \bar{X}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{\sum x_i}{\theta^2} < 0$$

• $\theta = \bar{X}$ is m.l.e for θ .

Ex: Let X_1, X_2, \dots, X_n be a rsn from normal distⁿ $N(\theta, 1)$, find the m.l.e for θ .

Sol:

$$X \sim N(\theta, 1)$$

$$f(x; \theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}, \quad -\infty < x < \infty$$

$$L(\theta) = \prod_{i=1}^n f(x_i; \theta) = \left(\frac{1}{\sqrt{2\pi}} \right)^n e^{-\frac{1}{2} \sum (x_i - \theta)^2}$$

$$\ln L(\theta) = n \ln \left(\frac{1}{\sqrt{2\pi}} \right) - \frac{1}{2} \sum (x_i - \theta)^2$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \text{zero} - \frac{2}{2} \sum (x_i - \theta) (-1) = \sum (x_i - \theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\sum (x_i - \theta) = 0 \quad \Rightarrow \quad \sum x_i - n\theta = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{\sum X_i}{n} = \bar{X}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -n < 0 \quad \Rightarrow \quad \therefore \hat{\theta} = \bar{X} \text{ is m.l.e for } \theta.$$

Ex: Let X_1, X_2, \dots, X_n be a rsn from Binomial distⁿ $\text{Bin}(m, \theta)$, find the m.l.e for θ .

Sol:

$$X \sim \text{Bin}(m, \theta)$$

$$f(x; \theta) = C_x^m \theta^x (1-\theta)^{m-x}, \quad x = 0, 1, \dots, m$$

$$L(\theta) = \prod_{i=1}^n f(x_i; \theta) = \prod_{i=1}^n C_{x_i}^m \theta^{\sum x_i} (1-\theta)^{\sum (m-x_i)}$$

$$\ln L(\theta) = \ln \prod_{i=1}^n C_{x_i}^m + \sum x_i \ln(\theta) + \sum (m-x_i) \ln(1-\theta)$$

$$\begin{aligned} \frac{\partial \ln L(\theta)}{\partial \theta} &= \text{zero} + \frac{\sum x_i}{\theta} + \frac{\sum (m-x_i)}{(1-\theta)} \times (-1) = \frac{\sum x_i}{\theta} - \frac{\sum (m-x_i)}{(1-\theta)} \\ &= \frac{\sum x_i(1-\theta) - \theta \sum (m-x_i)}{\theta(1-\theta)} = \frac{\sum x_i - \theta \sum x_i - nm\theta + \theta \sum x_i}{\theta(1-\theta)} = \frac{\sum x_i - nm\theta}{\theta(1-\theta)} \end{aligned}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\frac{\sum x_i - nm\theta}{\theta(1-\theta)} = 0 \quad \Rightarrow \quad \sum x_i - nm\theta = 0 \quad \Rightarrow \quad \sum x_i = nm\theta \quad \Rightarrow \quad \hat{\theta} = \frac{\sum X_i}{nm}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{-\sum x_i}{\theta^2} - \frac{\sum (m-x_i)}{(1-\theta)^2} = -\frac{\sum x_i}{\theta^2} - \frac{nm - \sum x_i}{(1-\theta)^2} < 0$$

$\therefore \hat{\theta}$ is m.l.e for θ .

Remarks:

- 1) The m.l.e. $\hat{\theta}$ is a function of the sufficient estimator.
- 2) The m.l.e. $\hat{\theta}$ is not always unbiased estimator for θ .

Invariance Property of the (m.l.e)

In a rsn from a distⁿ with p.d.f. $f(x;\theta)$, let $\hat{\theta}$ be a m.l.e. for the parameter θ , and $u(\theta)$ be a (one-to-one) function of θ , then $u(\hat{\theta})$ is a m.l.e. for $u(\theta)$.

Ex: In a rsn from exponential distⁿ $\text{Exp}(1/\theta)$, find the m.l.e for:

$$1) u_1(\theta) = \frac{1}{\theta} \quad 2) u_2(\theta) = \frac{\ln(\theta)}{\theta}$$

Sol:

$$X \sim \text{Exp}(1/\theta)$$

$$f(x;\theta) = \theta e^{-\theta x}, \quad x > 0$$

$$L(\theta) = \prod_{i=1}^n f(x_i; \theta) = \theta^n e^{-\theta \sum x_i}$$

$$\ln L(\theta) = n \ln(\theta) - \theta \sum x_i$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta} - \sum x_i, \quad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\frac{n}{\theta} - \sum x_i = 0 \quad \Rightarrow \quad \frac{n}{\theta} = \sum x_i \quad \Rightarrow \quad \hat{\theta} = \frac{n}{\sum X_i} = \frac{1}{\bar{X}}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{-n}{\theta^2} < 0 \quad \Rightarrow \therefore \hat{\theta} \text{ is m.l.e for } \theta.$$

$$1) u_1(\hat{\theta}) = \frac{1}{\hat{\theta}} = \frac{1}{\frac{1}{\bar{x}}} = \bar{x} \quad 2) u_2(\hat{\theta}) = \frac{\ln(\hat{\theta})}{\hat{\theta}} = \frac{\ln\left(\frac{1}{\bar{x}}\right)}{\frac{1}{\bar{x}}} = \bar{x} \ln\left(\frac{1}{\bar{x}}\right)$$

Ex: In a rsn from exponential distⁿ $\text{Exp}(\theta)$, find the m.l.e for θ :

Sol: $X \sim \text{Exp}(\theta)$

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0$$

$\because X$'s are indep.

$$L(\theta) = f(x_1, x_1, \dots, x_1; \theta) = \prod f(x_i; \theta)$$

$$= \frac{1}{\theta^n} e^{-\sum x_i/\theta}$$

$$\ln L(\theta) = -n \ln(\theta) - \frac{\sum x_i}{\theta}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2}, \quad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$-\frac{n}{\theta^2} + \frac{\sum x_i}{\theta^2} = 0$$

$$\frac{-n\theta + \sum x_i}{\theta^2} = 0$$

$$-n\theta + \sum x_i = 0$$

$$\sum x_i = n\theta \quad \theta_{m.l.e} = \frac{\sum X_i}{n} = \bar{X}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} \quad \} \times \theta$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -n + \frac{\sum x_i}{\theta}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{\sum x_i}{\theta^2} < 0$$

• $\hat{\theta} = \bar{X}$ is m.l.e for θ .

Ex: In a rsn from Geometric distⁿ Geo(θ), with p.d.f ; $f(x;\theta) = \theta(1 - \theta)^{x-1}$, $x = 1,2,\dots$, find the m.l.e for θ :

Sol: $X \sim \text{Geo}(\theta)$

$$f(x;\theta) = \theta(1 - \theta)^{x-1}, \quad x = 1,2,\dots$$

∴ X 's are indep.

$$L(\theta) = f(x_1, x_1, \dots, x_1; \theta) = \prod f(x_i; \theta)$$

$$= \theta^n (1 - \theta)^{\sum x_i - n}$$

$$\ln L(\theta) = n \ln(\theta) + (\sum x_i - n) \ln(1 - \theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta} - \frac{(\sum x_i - n)}{(1 - \theta)}, \quad \frac{\partial \ln L(\theta)}{\partial \theta} = 0$$

$$\frac{n}{\theta} - \frac{(\sum x_i - n)}{(1 - \theta)} = 0$$

$$\frac{n(1 - \hat{\theta}) - \hat{\theta}(\sum x_i - n)}{\hat{\theta}(1 - \hat{\theta})} = 0$$

$$n - n\hat{\theta} - \hat{\theta}\sum x_i + n\hat{\theta} = 0$$

$$n - \hat{\theta}\sum x_i = 0$$

$$\hat{\theta}\sum x_i = n \quad \hat{\theta}_{m.l.e} = \frac{n}{\sum x_i} = \frac{1}{\bar{X}}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{n}{\theta^2} - \frac{(\sum x_i - n)}{(1 - \theta)^2} < 0$$

• $\hat{\theta} = 1/\bar{X}$ is m.l.e for θ .

H.W: In a rsn from Geometric distⁿ Geo(θ), with p.d.f ; $f(x;\theta) = \theta(1 - \theta)^x$, $x = 0,1,2,\dots$, find the m.l.e for θ :

Ex: In a rsn taken from a distⁿ with p.d.f ; $f(x;\theta) = e^{-(x-\theta)}$, $\theta \leq x < \infty$, find the m.l.e for θ .

$$f(x;\theta) = e^{-(x-\theta)}$$

$$L(\theta) = \prod_{i=1}^n f(x_i;\theta) = e^{-\sum (x_i - \theta)}$$

$$\ln L(\theta) = - \sum (x_i - \theta) = - \sum x_i + n\theta$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \text{zero} + n = n$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0 \quad \Rightarrow n \neq 0 \quad (n: \text{sample size})$$

$$\theta \leq x_i \quad (y_1, y_2, \dots, y_n)$$

$$\theta \leq \text{Min}(X_i) \quad \Rightarrow \hat{\theta} = Y_1$$

$\theta^{\wedge} = Y_1$ is the m.l.e. for θ .

Ex: Let X_1, X_2, \dots, X_n be a rsn from normal distⁿ $N(\theta, \sigma^2)$, **1)** find m.l.e for parameters θ and σ^2 . **2)** If S^2 is m.l.e. for σ^2 , then find m.l.e. for σ .

Sol: 1)

$$X \sim N(\theta, \sigma^2)$$

$$f(x; \theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}, \quad -\infty < x < \infty$$

$$L(\theta, \sigma^2) = \prod_{i=1}^n f(x_i; \theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n e^{-\frac{1}{2\sigma^2} \sum (x_i - \theta)^2}$$

$$= (2\pi)^{-n/2} (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum (x_i - \theta)^2}$$

1) For θ ?

$$\ln L(\theta, \sigma^2) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum (x_i - \theta)^2$$

$$\frac{\partial \ln L(\theta, \sigma^2)}{\partial \theta} = \text{zero} - \text{zero} - \frac{2}{2\sigma^2} \sum (x_i - \theta) (-1) = \frac{\sum (x_i - \theta)}{\sigma^2} = \frac{\sum x_i - n\theta}{\sigma^2}$$

$$\frac{\partial \ln L(\theta, \sigma^2)}{\partial \theta} = 0$$

$$\frac{\sum x_i - n\theta}{\sigma^2} = 0 \quad \Rightarrow \sum x_i - n\theta = 0 \quad \Rightarrow \sum x_i = n\theta \quad \Rightarrow \hat{\theta}_{m.l.e} = \frac{\sum x_i}{n} = \bar{x}$$

$$\frac{\partial^2 \ln L(\theta, \sigma^2)}{\partial \theta^2} = \frac{-n}{\sigma^2} < 0$$

$\therefore \hat{\theta} = \bar{x}$ is m.l.e for θ .

2) For σ^2 ?

$$\frac{\partial \ln L(\theta, \sigma^2)}{\partial \sigma^2} = \text{zero} - \frac{n}{2\sigma^2} + \frac{2\sum (x_i - \theta)^2}{4\sigma^4} = -\frac{n}{2\sigma^2} + \frac{\sum (x_i - \theta)^2}{2\sigma^4}$$

$$= \frac{-n\sigma^2 + \sum (x_i - \theta)^2}{2\sigma^4} = 0$$

$$-n\sigma^2 + \sum (x_i - \theta)^2 = 0 \quad \Rightarrow n\sigma^2 = \sum (x_i - \theta)^2$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum (x_i - \hat{\theta})^2}{n} = \frac{\sum (X_i - \bar{X})^2}{n} = S^2$$

$$\therefore \frac{\partial \ln L(\theta, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum (x_i - \theta)^2}{2\sigma^4} \quad \} \times 2\sigma^2$$

$$= -n + \frac{\sum (x_i - \theta)^2}{\sigma^2}$$

$$\frac{\partial^2 \ln L(\theta, \sigma^2)}{\partial (\sigma^2)^2} = \text{zero} - \frac{\sum (x_i - \theta)^2}{\sigma^4} = -\frac{\sum (x_i - \theta)^2}{\sigma^4} < 0$$

$\therefore S^2$ is m.l.e for σ^2

2)

$$u(\sigma^2) = u(\hat{\sigma}^2)$$

$$u(\sigma^2) = \sqrt{\sigma^2} = \sigma$$

$$u(\hat{\sigma}^2 = S^2) = \sqrt{S^2} = \sqrt{\frac{\sum (X_i - \bar{X})^2}{n}} = S$$

$\therefore S$ is m.l.e. for σ

Second: Moments Estimation Method (MEM)

Let X_1, X_2, \dots, X_n be a rsn from a distⁿ with a p.d.f. $f(x; \theta)$, the average value of the k^{th} powers of (X_1, X_2, \dots, X_n) ; $m_k = \frac{\sum X_i^k}{n}$ is the k^{th} sample moment, $M_k = E(X^k)$ is the k^{th} population moment about origin. The moment's method estimator is the value of the unknown parameter $\hat{\theta}$ that makes:

$$m_k = M_k$$

Ex: Let X_1, X_2, \dots, X_n be a rsn from normal distⁿ $N(\theta, \sigma^2)$, estimate the parameters θ and σ^2 using moment method.

Sol:

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n}, \quad M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} \Rightarrow M_1 = E(X) = \theta$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \theta \Rightarrow \therefore \hat{\theta} = \bar{X}$$

$$m_2 = \frac{\sum X_i^2}{n} \Rightarrow M_2 = E(X^2)$$

$$M_2 = E(X^2) = V(X) + (E(X))^2 = \sigma^2 + \theta^2$$

$$m_2 = M_2$$

$$\frac{\sum X_i^2}{n} = \sigma^2 + \bar{X}^2$$

$$\therefore \hat{\sigma}^2 = \frac{\sum X_i^2}{n} - \bar{X}^2$$

Ex: In a rsn from a dist^n with p.d.f.; $f(x; \theta) = (\theta + 1) x^\theta$, $0 < x < 1$, estimate the parameter θ using moment method.

Sol:

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n}, \quad M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n}$$

$$M_1 = E(X)$$

$$E(X) = \int_0^1 x f(x; \theta) dx = \int_0^1 x (\theta + 1) x^\theta dx = \int_0^1 (\theta + 1) x^{\theta+1} dx = (\theta + 1) \frac{x^{\theta+2}}{\theta + 2} \Big|_0^1 = \frac{\theta + 1}{\theta + 2}$$

$$m_1 = M_1$$

$$\begin{aligned} \frac{\sum X_i}{n} = \frac{\theta + 1}{\theta + 2} &\Rightarrow \bar{X} = \frac{\theta + 1}{\theta + 2} \Rightarrow (\theta + 2) \bar{X} = \theta + 1 \Rightarrow \theta \bar{X} + 2 \bar{X} = \theta + 1 \\ &\Rightarrow \theta \bar{X} - \theta = 1 - 2 \bar{X} \Rightarrow \theta(\bar{X} - 1) = 1 - 2 \bar{X} \\ &\Rightarrow \therefore \hat{\theta} = \frac{1 - 2 \bar{X}}{\bar{X} - 1} \end{aligned}$$

Ex: Estimate the parameters of $\Gamma(\alpha, 1/\theta)$, using moment method.

Sol:

When $X \sim \Gamma(\alpha, \beta)$, $E(X) = \alpha \beta$, $V(X) = \alpha \beta^2$

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n}, \quad M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} \Rightarrow M_1 = E(X) = \alpha \beta = \frac{\alpha}{\theta}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{\alpha}{\theta} \Rightarrow \therefore \bar{X} = \frac{\alpha}{\theta} \quad \dots (1)$$

$$k = 2$$

$$m_2 = \frac{\sum X_i^2}{n} \Rightarrow M_2 = E(X^2)$$

$$M_2 = E(X^2) = V(X) + (E(X))^2 = \frac{\alpha}{\theta^2} + \frac{\alpha^2}{\theta^2} \quad \dots (2)$$

$$m_2 = M_2$$

$$\frac{\sum X_i^2}{n} = \frac{\alpha}{\theta^2} + \frac{\alpha^2}{\theta^2}, \quad \text{put (1) in (2)}$$

$$\frac{\sum X_i^2}{n} = \frac{\bar{X}}{\theta} + \bar{X}^2 \Rightarrow \frac{\sum X_i^2}{n} - \bar{X}^2 = \frac{\bar{X}}{\theta} \Rightarrow S^2 = \frac{\bar{X}}{\theta} \Rightarrow \hat{\theta} = \frac{\bar{X}}{S^2} \dots (3)$$

From (1)

$$\bar{X} = \frac{\alpha}{\theta} \Rightarrow \alpha = \theta \bar{X} \quad \text{put (3) in (1)}$$

$$\alpha = \frac{\bar{X}}{S^2} \bar{X} = \frac{\bar{X}^2}{S^2}$$

$$\Rightarrow \therefore \hat{\alpha}_{moment} = \frac{\bar{X}^2}{S^2}, \quad \hat{\theta}_{moment} = \frac{\bar{X}}{S^2}$$

Ex: Estimate the parameter by using moment method for:

1) Ber(θ). **2)** Exp($1/\theta$). **3)** Geo(θ).

Sol:

1) $X \sim \text{Ber}(\theta)$, $E(X) = \theta$

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n}, \quad M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} = \bar{X} \Rightarrow M_1 = E(X) = \theta$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \theta \Rightarrow \therefore \bar{X} = \theta, \quad \therefore \hat{\theta}_{moment} = \bar{X}$$

2) $X \sim \text{Exp}(1/\theta)$, $E(X) = 1/\theta$

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n}, \quad M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} = \bar{X} \Rightarrow M_1 = E(X) = \frac{1}{\theta}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{1}{\theta} \Rightarrow \therefore \bar{X} = \frac{1}{\theta}, \quad \therefore \hat{\theta}_{moment} = \frac{1}{\bar{X}}$$

3) $X \sim \text{Geo}(\theta)$, $E(X) = \frac{1-\theta}{\theta}$

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n}, \quad M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} = \bar{X} \Rightarrow M_1 = E(X) = \frac{1-\theta}{\theta}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{1-\theta}{\theta}$$

$$\Rightarrow \therefore \bar{X} = \frac{1-\theta}{\theta} \Rightarrow \bar{X} \theta = 1-\theta \Rightarrow \bar{X} \theta + \theta = 1 \Rightarrow \theta(\bar{X} + 1) = 1 \Rightarrow \therefore \hat{\theta}_{moment} = \frac{1}{\bar{X} + 1}$$

Ex: Find an estimate the parameter θ from; $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, by using moment method.

Sol:

$$m_k = M_k$$

$$m_k = \frac{\sum X_i^k}{n} \quad , \quad M_k = E(X^k)$$

$$m_1 = \frac{\sum X_i}{n} = \bar{X}$$

$$M_1 = E(X) = \int_0^1 x f(x;\theta) dx = \int_0^1 x \theta x^{\theta-1} dx = \int_0^1 \theta x^\theta dx = \theta \left. \frac{x^{\theta+1}}{\theta+1} \right|_0^1 = \frac{\theta}{\theta+1}$$

$$m_1 = M_1$$

$$\frac{\sum X_i}{n} = \frac{\theta}{\theta+1} \Rightarrow \therefore \bar{X} = \frac{\theta}{\theta+1} \Rightarrow (\theta+1)\bar{X} = \theta \Rightarrow \theta\bar{X} + \bar{X} = \theta \Rightarrow \theta\bar{X} - \theta = -\bar{X} \\ \Rightarrow \theta(\bar{X} - 1) = -\bar{X}$$

$$\therefore \hat{\theta}_{moment} = \frac{\bar{X}}{\bar{X} - 1}$$

Third: Minimum Variance Method (MVM)

Let $L(\theta)$ be the likelihood function of a rsn with p.d.f. $f(x;\theta)$, then the parameter θ has minimum variance unbiased estimator (m.v.u.e.) if it is possible to express $\left(\frac{\partial}{\partial \theta} \ln L(\theta)\right)$ in the following form;

$$\frac{\partial}{\partial \theta} \ln L(\theta) = \frac{\hat{\theta} - \theta}{V(\hat{\theta})}$$

Where; $\hat{\theta}$: is (m.v.e.) , $V(\hat{\theta})$: is variance of $\hat{\theta}$.

Ex: In a rsn, find m.v.e. for the parameters of; **1)** Ber(θ). **2)** N(θ, σ^2).

Sol:

1) $X \sim Ber(\theta)$

$$f(x;\theta) = \theta^x (1-\theta)^{1-x} \quad , \quad x = 0,1$$

$$L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \theta^{\sum x_i} (1-\theta)^{n - \sum x_i}$$

$$\ln L(\theta) = \sum x_i \ln(\theta) + (n - \sum x_i) \ln(1-\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{(1-\theta)} = \frac{(1-\theta)\sum x_i - \theta(n - \sum x_i)}{\theta(1-\theta)} = \frac{\sum x_i - \theta \sum x_i - n\theta + \theta \sum x_i}{\theta(1-\theta)}$$

$$= \frac{\sum x_i - n\theta}{\theta(1-\theta)} \quad (\div n)$$

$$= \frac{\bar{x} - \theta}{\theta(1-\theta)} = \frac{\hat{\theta} - \theta}{V(\hat{\theta})} \quad , \quad \hat{\theta} = \bar{X} \quad , \quad V(\hat{\theta}) = V(\bar{X}) = \frac{V(X)}{n} = \frac{\theta(1-\theta)}{n}$$

$\therefore \bar{X}$ is m.v.e. for θ .

$$2) X \sim N(\theta, \sigma^2)$$

$$f(x; \theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}, \quad -\infty < x < \infty$$

$$L(\theta, \sigma^2) = \prod_{i=1}^n f(x_i; \theta) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum (x_i - \theta)^2}$$

$$\ln L(\theta, \sigma^2) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum (x_i - \theta)^2$$

$$\frac{\partial \ln L(\theta, \sigma^2)}{\partial \theta} = \text{zero} - \frac{2}{2\sigma^2} \sum (x_i - \theta) (-1) = \frac{\sum (x_i - \theta)}{\sigma^2} = \frac{\sum x_i - n\theta}{\sigma^2} \quad \} \div n$$

$$\Rightarrow \frac{\bar{x} - \theta}{\sigma^2/n} = \frac{\hat{\theta} - \theta}{V(\hat{\theta})} \quad \Rightarrow \quad \hat{\theta} = \bar{X} \quad \Rightarrow \quad V(\hat{\theta}) = \frac{\sigma^2}{n}$$

$$\begin{aligned} \frac{\partial \ln L(\theta, \sigma^2)}{\partial \sigma^2} &= -\frac{n}{2} \cdot \frac{2\pi}{2\pi\sigma^2} + \frac{2\sum (x_i - \theta)^2}{4\sigma^4} = -\frac{n}{2\sigma^2} + \frac{\sum (x_i - \theta)^2}{2\sigma^4} \\ &= \frac{-n\sigma^2 + \sum (x_i - \theta)^2}{2\sigma^4} \quad \} \div n \end{aligned}$$

$$= \frac{\frac{\sum (x_i - \theta)^2}{n} - \sigma^2}{2\sigma^4} = \frac{S^2 - \sigma^2}{V(S^2)} = \frac{\hat{\sigma}^2 - \sigma^2}{V(\hat{\sigma}^2)}$$

$$\therefore \hat{\theta} = \bar{X}, \quad S^2 = \frac{\sum (X_i - \bar{X})^2}{n}, \quad V(S^2) = \frac{2\sigma^4}{n}$$

Fourth: Bayesian Estimation Method (BEM)

Philosophy: Observed data X is fixed, and the unknown generating parameter θ is random. (Certainty about θ depends on both empirical information X and prior knowledge about θ).

In Bayesian estimation method the parameters treats as a random variable with prior probability $p(\theta)$, or we have prior informative about the parameter θ .

Let A and B be two events, then the conditional probability of A given B is;

$$p(A | B) = \frac{p(A B)}{p(B)} = \frac{p(B | A) p(A)}{p(B)}$$

Let; $A = \theta$ and $B = x$, then in a rsn with p.d.f. $f(x; \theta)$ and prior probability $p(\theta)$;

$$p(\theta | x) = \frac{p(x | \theta) p(\theta)}{p(x)}$$

$p(x)$ does not contain θ , we can write it as;

$$\begin{aligned} p(\theta | x) &\propto p(x | \theta) p(\theta) \\ &\propto L(\theta) p(\theta) \end{aligned}$$

Where;

$p(\theta | x)$: is called posterior probability and Bayes estimator denote $\hat{\theta}_{Bayes}$ is the mean of posterior probability $E(\theta | X)$.

$L(\theta)$: is likelihood function.

$p(\theta)$: is prior probability.

We have two types of prior probability:

1) Non Informative prior probability (Jeffery's rule).

2) Informative prior probability.

First: Non Informative prior probability (Jeffery's rule)

It is proportional to the square root of Fisher information;

$$p(\theta) \propto (I_s(\theta))^{1/2} \quad , \quad I_s = F.I.$$

Ex: Find Bayes estimator for parameter of; **1)** $Ber(\theta)$. **2)** $Poisson(\theta)$, using non informative prior probability.

Sol:

1) $X \sim Ber(\theta)$

$$f(x; \theta) = \theta^x (1 - \theta)^{1-x}$$

$$p(\theta | x) \propto L(\theta) p(\theta)$$

$$L(\theta) = \theta^{\sum x_i} (1 - \theta)^{n - \sum x_i}$$

$$p(\theta) \propto (I_s(\theta))^{1/2} \quad , \quad F.I = - E\left(\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2}\right)$$

$$\ln f(x; \theta) = x \ln(\theta) + (1 - x) \ln(1 - \theta)$$

$$\frac{\partial \ln f(x; \theta)}{\partial \theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2} = \frac{-x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$$

$$\begin{aligned} F.I. - E\left(\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2}\right) &= \frac{E(X)}{\theta^2} + \frac{E(1-X)}{(1-\theta)^2} \\ &= \frac{E(X)}{\theta^2} + \frac{1-E(X)}{(1-\theta)^2} = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)} \end{aligned}$$

$$\begin{aligned} p(\theta) &\propto \left(\frac{1}{\theta(1-\theta)}\right)^{1/2} \\ &\propto \theta^{-1/2} (1-\theta)^{-1/2} \end{aligned}$$

$$\begin{aligned}
p(\theta | x) &\propto L(\theta) p(\theta) \\
&\propto \theta^{\sum x_i} e^{n - \sum x_i} \theta^{-1/2} (1 - \theta)^{-1/2} \\
&\propto \theta^{\sum x_i - \frac{1}{2}} (1 - \theta)^{n - \sum x_i - \frac{1}{2}}
\end{aligned}$$

$$\alpha - 1 = \sum x_i - \frac{1}{2} \quad \Rightarrow \quad \alpha = \sum x_i + \frac{1}{2}$$

$$\beta - 1 = n - \sum x_i - \frac{1}{2} \quad \Rightarrow \quad \beta = n - \sum x_i + \frac{1}{2}$$

$$p(\theta | x) \sim \text{Beta}\left(\alpha = \sum x_i + \frac{1}{2}, \beta = n - \sum x_i + \frac{1}{2}\right)$$

When; $X \sim \text{Beta}(\alpha, \beta)$,

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad , \quad E(X) = \frac{\alpha}{\alpha + \beta}$$

then the complete p.d.f. of the posterior probability is;

$$p(\theta | x) = \frac{\Gamma(n+1)}{\Gamma\left(\sum x_i + \frac{1}{2}\right) \Gamma\left(n - \sum x_i + \frac{1}{2}\right)} \theta^{\sum x_i - \frac{1}{2}} (1 - \theta)^{n - \sum x_i - \frac{1}{2}}$$

$$\begin{aligned}
\therefore E(\theta | X_1, \dots, X_n) &= \hat{\theta}_{\text{Bayes}} = \frac{\alpha}{\alpha + \beta} \\
&= \frac{\sum X_i + \frac{1}{2}}{n + 1} = \frac{\sum X_i}{n + 1} + \frac{1}{2n + 2}
\end{aligned}$$

2) $X \sim \text{Poi}(\theta)$

$$f(x; \theta) = \frac{e^{-\theta} \theta^x}{x!} \quad , \quad x = 0, 1, \dots$$

$$\ln f(x; \theta) = -\theta + x \ln(\theta) - \ln(x!)$$

$$\frac{\partial \ln f(x; \theta)}{\partial \theta} = -1 + \frac{x}{\theta}$$

$$\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2} = -\frac{x}{\theta^2}$$

$$-E\left(\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2}\right) = \frac{E(X)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$$p(\theta) \propto (I_s(\theta))^{1/2}$$

$$\propto \left(\frac{1}{\theta}\right)^{1/2} = \theta^{-1/2}$$

$$L(\theta) = \frac{e^{-n\theta} \theta^{\sum x_i}}{\left(\prod_{i=1}^n x_i\right)!}$$

$$L(\theta) \propto e^{-n\theta} \theta^{\sum x_i}$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto L(\theta) p(\theta)$$

$$\propto e^{-n\theta} \theta^{\sum x_i} \theta^{-1/2}$$

$$\propto e^{-n\theta} \theta^{\sum x_i - \frac{1}{2}}$$

$$p(\theta | x_1, x_2, \dots, x_n) \sim \Gamma(\alpha = \sum x_i + \frac{1}{2}, \beta = n)$$

$$\text{when } X \sim \Gamma(\alpha, \beta), \quad f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad E(X) = \frac{\alpha}{\beta}$$

$$\alpha - 1 = \sum x_i - \frac{1}{2} \Rightarrow \alpha = \sum x_i + \frac{1}{2}$$

$$\beta = n$$

$$p(\theta | x_1, x_2, \dots, x_n) = \frac{n^{\sum x_i + \frac{1}{2}}}{\Gamma(\sum x_i + \frac{1}{2})} \theta^{\sum x_i - \frac{1}{2}} e^{-n\theta}$$

$$\therefore \hat{\theta}_{\text{Bayes}} = E(\theta | x_1, x_2, \dots, x_n) = \frac{\sum X_i + \frac{1}{2}}{n} = \bar{X} + \frac{1}{2n}$$

Ex: Find Bayes estimator for parameters of; **1)** $\text{Exp}(1/\theta)$, **2)** $N(\theta, \sigma^2)$, using non informative prior probability.

Sol:

$$1) X \sim \text{Exp}(1/\theta)$$

$$f(x; \theta) = \theta e^{-\theta x}, \quad x > 0$$

$$p(\theta | x) \propto L(\theta) p(\theta)$$

$$L(\theta) = \theta^n e^{-\theta \sum x_i}$$

$$p(\theta) \propto (I_s(\theta))^{1/2}, \quad F.I = -E\left(\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2}\right)$$

$$\ln f(x; \theta) = \ln(\theta) - \theta x$$

$$\frac{\partial \ln f(x; \theta)}{\partial \theta} = \frac{1}{\theta} - x$$

$$\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2} = \frac{-1}{\theta^2}$$

$$F.I = - E \left(\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2} \right) = \frac{1}{\theta^2}$$

$$p(\theta) \propto \left(\frac{1}{\theta^2} \right)^{1/2}$$

$$\propto \theta^{-1}$$

$$p(\theta | x) \propto L(\theta) p(\theta)$$

$$\propto \theta^n e^{-\theta \sum x_i} \theta^{-1}$$

$$\propto \theta^{n-1} e^{-\theta \sum x_i}$$

$$p(\theta | x) \sim \Gamma(\alpha = n, \beta = \sum x_i)$$

$$\text{when; } f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \Rightarrow E(X) = \frac{\alpha}{\beta}$$

then the complete p.d.f. of the posterior probability is;

$$p(\theta | x) = \frac{(\sum x_i)^n}{\Gamma(n)} \theta^{n-1} e^{-\theta \sum x_i}$$

$$\therefore E(\theta | X_1, \dots, X_n) = \hat{\theta}_{\text{Bayes}} = \frac{\alpha}{\beta} = \frac{n}{\sum X_i} = \frac{1}{\bar{X}}$$

$$2) X \sim N(\theta, \sigma^2)$$

$$f(x; \theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$L(\theta, \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum (x_i - \theta)^2}$$

1) For θ

$$p(\theta | x_1, x_2, \dots, x_n) \propto L(\theta) p(\theta)$$

$$L(\theta) \propto e^{-\frac{1}{2\sigma^2}\sum (x_i - \theta)^2}$$

$$\propto e^{-\frac{1}{2\sigma^2}\sum (x_i - \theta + \bar{x} - \bar{x})^2} \quad \} \mp \bar{x}$$

$$\propto e^{-\frac{1}{2\sigma^2}\sum (x_i - \bar{x})^2} e^{-\frac{1}{2\sigma^2}n(\theta - \bar{x})^2}$$

$$\therefore L(\theta) \propto e^{-\frac{1}{2\sigma^2}n(\theta - \bar{x})^2}$$

$$\ln f(x; \theta, \sigma^2) = -\frac{1}{2} \ln(2\pi\sigma^2) - \frac{(x-\theta)^2}{2\sigma^2}$$

$$\frac{\partial \ln f(x; \theta, \sigma^2)}{\partial \theta} = \text{zero} - \frac{2(x-\theta)(-1)}{2\sigma^2} = \frac{(x-\theta)}{\sigma^2}$$

$$\frac{\partial^2 \ln f(x; \theta, \sigma^2)}{\partial \theta^2} = -\frac{1}{\sigma^2}$$

$$- E \left(\frac{\partial^2 \ln f(x; \theta, \sigma^2)}{\partial \theta^2} \right) = \frac{1}{\sigma^2} = F.I.$$

$$\begin{aligned} p(\theta) &\propto (I_s(\theta))^{1/2} \\ &\propto \left(\frac{1}{\sigma^2} \right)^{1/2} \\ &\propto \text{Cons tan } t \\ &\propto 1 \end{aligned}$$

$$\begin{aligned} p(\theta | x_1, x_2, \dots, x_n) &\propto L(\theta) p(\theta) \\ &\propto e^{-\frac{1}{2\sigma^2} n (\theta - \bar{x})^2} \times (1) \\ &\propto e^{-\frac{1}{2\sigma^2} n (\theta - \bar{x})^2} \end{aligned}$$

$$p(\theta | x_1, x_2, \dots, x_n) \sim N(\bar{X}, \frac{\sigma^2}{n})$$

$$\therefore \text{mean} = \bar{X} \quad , \quad \text{variance} = \frac{\sigma^2}{n}$$

$$\therefore p.d.f. \Rightarrow p(\theta | x_1, x_2, \dots, x_n) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{n}}} e^{-\frac{n}{2\sigma^2} (\theta - \bar{x})^2}$$

$$\therefore \hat{\theta}_{\text{Bayes}} = E(p(\theta | x_1, x_2, \dots, x_n)) = \bar{X}$$

2) For σ^2

$$p(\sigma^2 | x_1, x_2, \dots, x_n) \propto L(\sigma^2) p(\sigma^2)$$

$$L(\sigma^2) \propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum (x_i - \theta)^2}$$

$$p(\sigma^2) \propto (I_s(\sigma^2))^{1/2}$$

$$\ln f(x, \sigma^2) = -\frac{1}{2} \ln(2\pi\sigma^2) - \frac{(x - \theta)^2}{2\sigma^2}$$

$$\frac{\partial \ln f(x; \sigma^2)}{\partial \sigma^2} = -\frac{1}{2} \frac{2\pi}{2\pi\sigma^2} + \frac{(x_i - \theta)^2 (2)}{4\sigma^4} = -\frac{1}{2\sigma^2} + \frac{(x_i - \theta)^2}{2\sigma^4}$$

$$\frac{\partial^2 \ln f(x; \sigma^2)}{\partial (\sigma^2)^2} = \frac{2}{4(\sigma^2)^2} - \frac{(x_i - \theta)^2 (4\sigma^2)}{4(\sigma^2)^4} = \frac{1}{2(\sigma^2)^2} - \frac{(x_i - \theta)^2}{(\sigma^2)^3}$$

$$\begin{aligned}
-E\left(\frac{\partial^2 \ln f(x; \sigma^2)}{\partial (\sigma^2)^2}\right) &= \frac{-1}{2(\sigma^2)^2} + \frac{E(x_i - \theta)^2}{(\sigma^2)^3} = \frac{-1}{2(\sigma^2)^2} + \frac{\sigma^2}{(\sigma^2)^3} = \frac{-1}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^2} \\
&= \frac{-1+2}{2(\sigma^2)^2} = \frac{1}{2(\sigma^2)^2} = F.I.
\end{aligned}$$

$$\begin{aligned}
p(\sigma^2) &\propto (I_s(\theta))^{1/2} \\
&\propto \left(\frac{1}{(\sigma^2)^2}\right)^{1/2} \\
&\propto (\sigma^2)^{-1}
\end{aligned}$$

$$\begin{aligned}
p(\sigma^2 | x_1, x_2, \dots, x_n) &\propto L(\sigma^2) p(\sigma^2) \\
&\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum (x_i - \theta)^2} \times (\sigma^2)^{-1} \\
&\propto (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\frac{1}{2\sigma^2} \sum (x_i - \theta)^2}
\end{aligned}$$

$$p(\sigma^2 | x_1, x_2, \dots, x_n) \sim \Gamma^{-1}\left(\alpha = \frac{n}{2}, \beta = \frac{\sum (x_i - \theta)^2}{2}\right)$$

when $X \sim \Gamma^{-1}(\alpha, \beta)$, $f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x}$, $E(X) = \frac{\beta}{\alpha-1}$

$$\therefore p.d.f. \Rightarrow p(\sigma^2 | x_1, x_2, \dots, x_n) = \frac{\left(\frac{\sum (x_i - \theta)^2}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\left(\frac{\sum (x_i - \theta)^2}{2\sigma^2}\right)}$$

$$\therefore \hat{\sigma}_{Bayes}^2 = E(\sigma^2 | x_1, x_2, \dots, x_n) = \frac{\frac{\sum (X_i - \theta)^2}{2}}{\frac{n}{2} - 1} = \frac{\sum (X_i - \theta)^2}{n - 2}$$

Second: Informative prior probability

The form of prior probability for parameters of some distⁿ as follows:

ID	Probability Distribution	Informative Prior Probability
1	Bernoulli ~ Ber(θ)	Beta ~ Beta(α_o, β_o)
2	Binomial ~ Bin(n, θ)	Beta ~ Beta(α_o, β_o)
3	Geometric ~ Geo(θ)	Beta ~ Beta(α_o, β_o)
4	Poisson ~ Poi(θ)	Gamma ~ $\Gamma(\alpha_o, \beta_o)$
5	Exponential ~ Exp($1/\theta$)	Gamma ~ $\Gamma(\alpha_o, \beta_o)$
6	Exponential ~ Exp(θ)	Inverse Gamma ~ $\Gamma^{-1}(\alpha_o, \beta_o)$
7	Normal ~ N(θ, σ^2) (θ known)	Inverse Gamma ~ $\Gamma^{-1}(\alpha_o/2, \beta_o/2)$
8	Normal ~ N(θ, σ^2) (σ^2 known)	Normal ~ N(θ_o, σ_o^2)

Ex: Estimate the parameters of; **1)** Geo(θ). **2)** Poisson (θ). **3)** Exp(θ). **4)** N(θ, σ^2) (θ known) and (σ^2 known), using Bayesian informative prior probability.

Sol:

1) $X \sim Geo(\theta)$

$$f(x; \theta) = \theta(1 - \theta)^x$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto L(\theta) p(\theta)$$

$$L(\theta) = \theta^n (1 - \theta)^{\sum x_i}$$

$$p(\theta) \sim Beta(\alpha_o, \beta_o)$$

$$p(\theta) = \frac{\Gamma(\alpha_o + \beta_o)}{\Gamma(\alpha_o) \Gamma(\beta_o)} \theta^{\alpha_o - 1} (1 - \theta)^{\beta_o - 1}$$

$$\propto \theta^{\alpha_o - 1} (1 - \theta)^{\beta_o - 1}$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto \theta^n (1 - \theta)^{\sum x_i} \theta^{\alpha_o - 1} (1 - \theta)^{\beta_o - 1}$$

$$\propto \theta^{(\alpha_o + n) - 1} (1 - \theta)^{(\sum x_i + \beta_o) - 1}$$

$$p(\theta | x_1, x_2, \dots, x_n) \sim Beta(\alpha = \alpha_o + n, \beta = \sum x_i + \beta_o)$$

The complete p.d.f. of the posterior probability is;

$$\therefore p(\theta | x_1, x_2, \dots, x_n) = \frac{\Gamma(\alpha_o + n + \sum x_i + \beta_o)}{\Gamma(\alpha_o + n) \Gamma(\sum x_i + \beta_o)} \theta^{(\alpha_o + n) - 1} (1 - \theta)^{(\sum x_i + \beta_o) - 1}$$

$$\therefore E(\theta | X) = \hat{\theta}_{Bayes} = \frac{\alpha}{\alpha + \beta} = \frac{\alpha_o + n}{\alpha_o + n + \sum X_i + \beta_o}$$

2) $X \sim Poi(\theta)$

$$f(x; \theta) = \frac{e^{-\theta} \theta^x}{x!}, \quad x = 0, 1, \dots$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto L(\theta) p(\theta)$$

$$L(\theta) = \frac{e^{-n\theta} \theta^{\sum x_i}}{(\prod_{i=1}^n x_i)!}$$

$$\propto e^{-n\theta} \theta^{\sum x_i}$$

$$p(\theta) \sim Gamma(\alpha_o, \beta_o)$$

$$p(\theta) = \frac{\beta_o^{\alpha_o}}{\Gamma(\alpha_o)} \theta^{\alpha_o - 1} e^{-\beta_o \theta}$$

$$\propto \theta^{\alpha_o - 1} e^{-\beta_o \theta}$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto e^{-n\theta} \theta^{\sum x_i} \theta^{\alpha_o - 1} e^{-\beta_o \theta}$$

$$\propto \theta^{(\sum x_i + \alpha_o) - 1} e^{-(\beta_o + n)\theta}$$

$$p(\theta | x_1, x_2, \dots, x_n) \sim \Gamma(\alpha = \sum x_i + \alpha_o, \beta = \beta_o + n)$$

The complete p.d.f. of the posterior probability is;

$$p(\theta | x_1, x_2, \dots, x_n) = \frac{(\beta_o + n)^{\sum x_i + \alpha_o}}{\Gamma(\sum x_i + \alpha_o)} \theta^{\sum x_i + \alpha_o - 1} e^{-\theta(\beta_o + n)}$$

$$\therefore E(\theta | X) = \hat{\theta}_{Bayes} = \frac{\alpha}{\beta} = \frac{\sum X_i + \alpha_o}{\beta_o + n}$$

3) $X \sim \text{Exp}(\theta)$

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto L(\theta) p(\theta)$$

$$L(\theta) = \theta^{-n} \theta^{\sum x_i / \theta}$$

$$p(\theta) \sim \Gamma^{-1}(\alpha_o, \beta_o)$$

$$p(\theta) = \frac{\beta_o^{\alpha_o}}{\Gamma(\alpha_o)} \theta^{-(\alpha_o + 1)} e^{-\beta_o / \theta}$$

$$\propto \theta^{-(\alpha_o + 1)} e^{-\beta_o / \theta}$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto \theta^{-n} e^{-\sum x_i / \theta} \theta^{-(\alpha_o + 1)} e^{-\beta_o / \theta}$$

$$\propto \theta^{-((n + \alpha_o) + 1)} e^{-(\sum x_i + \beta_o) / \theta}$$

$$p(\theta | x_1, x_2, \dots, x_n) \sim \Gamma^{-1}(\alpha = n + \alpha_o, \beta = \sum x_i + \beta_o)$$

The complete p.d.f. of the posterior probability is;

$$p(\theta | x_1, x_2, \dots, x_n) = \frac{(\sum x_i + \beta_o)^{n + \alpha_o}}{\Gamma(n + \alpha_o)} \theta^{-((n + \alpha_o) + 1)} e^{-(\sum x_i + \beta_o) / \theta}$$

$$\therefore E(\theta | X_1, X_2, \dots, X_n) = \hat{\theta}_{Bayes} = \frac{\beta}{\alpha - 1} = \frac{\sum X_i + \beta_o}{n + \alpha_o - 1}$$

4) $X \sim N(\theta, \sigma^2)$, When $(\theta \text{ known})$

$$f(x; \theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x - \theta)^2}$$

$$p(\sigma^2 | x_1, x_2, \dots, x_n) \propto L(\sigma^2) p(\sigma^2)$$

$$L(\sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum (x_i - \theta)^2}$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{nS^2}{2\sigma^2}}$$

$$p(\sigma^2) \sim \Gamma^{-1}\left(\frac{\alpha_o}{2}, \frac{\beta_o}{2}\right)$$

$$p(\sigma^2) = \frac{\left(\frac{\beta_o}{2}\right)^{\alpha_o/2}}{\Gamma\left(\frac{\alpha_o}{2}\right)} (\sigma^2)^{-\left(\frac{\alpha_o}{2} + 1\right)} e^{-\left(\frac{\beta_o}{2\sigma^2}\right)}$$

$$\propto (\sigma^2)^{-\left(\frac{\alpha_o}{2} + 1\right)} e^{-\left(\frac{\beta_o}{2\sigma^2}\right)}$$

$$p(\sigma^2 | x_1, x_2, \dots, x_n) \propto (\sigma^2)^{-n/2} e^{-\frac{n S^2}{2\sigma^2}} (\sigma^2)^{-\left(\frac{\alpha_o}{2} + 1\right)} e^{-\left(\frac{\beta_o}{2\sigma^2}\right)}$$

$$\propto (\sigma^2)^{-\left(\frac{\alpha_o + n}{2} + 1\right)} e^{-\left(\frac{\beta_o + n S^2}{2\sigma^2}\right)}$$

$$p(\sigma^2 | x_1, x_2, \dots, x_n) \sim \Gamma^{-1}\left(\alpha = \frac{\alpha_o + n}{2}, \beta = \frac{\beta_o + n S^2}{2}\right)$$

$$\therefore p(\sigma^2 | x_1, x_2, \dots, x_n) = \frac{\left(\frac{\beta_o + n S^2}{2}\right)^{\frac{\alpha_o + n}{2}}}{\Gamma\left(\frac{\alpha_o + n}{2}\right)} (\sigma^2)^{-\left(\frac{\alpha_o + n}{2} + 1\right)} e^{-\left(\frac{\beta_o + n S^2}{2\sigma^2}\right)}$$

$$\therefore E(\sigma^2 | X) = \hat{\sigma}_{Bayes}^2 = \frac{\beta}{\alpha - 1} = \frac{(\beta_o + n S^2) / 2}{\frac{\alpha_o + n}{2} - 1} = \frac{\beta_o + n S^2}{\alpha_o + n - 2}$$

$X \sim N(\theta, \sigma^2)$ When σ^2 known.

$$p(\theta | x_1, x_2, \dots, x_n) \propto L(\theta) p(\theta)$$

$$L(\theta) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i - \bar{x})^2} e^{-\frac{1}{2\sigma^2}n(\theta - \bar{x})^2}, \quad \bar{x}$$

$$\propto e^{-\frac{1}{2\sigma^2}n(\theta - \bar{x})^2}$$

$$p(\theta) \sim N(\theta_o, \sigma_o^2)$$

$$p(\theta) = \frac{1}{\sqrt{2\pi\sigma_o^2}} e^{-\frac{1}{2\sigma_o^2}(\theta - \theta_o)^2}$$

$$\propto e^{-\frac{1}{2\sigma_o^2}(\theta - \theta_o)^2}$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto L(\theta, \sigma^2) p(\theta)$$

$$\propto e^{-\frac{1}{2\sigma^2}n(\theta - \bar{x})^2} e^{-\frac{1}{2\sigma_o^2}(\theta - \theta_o)^2}$$

$$\propto e^{-\frac{1}{2} \left(\frac{n}{\sigma^2} (\theta - \bar{x})^2 + \frac{1}{\sigma_o^2} (\theta - \theta_o)^2 \right)}$$

$$A(Z - a)^2 + B(Z - b)^2 = D(Z - c)^2$$

$$A = \frac{n}{\sigma^2}, \quad a = \bar{x}, \quad Z = \theta \text{ (variable)}$$

$$B = \frac{1}{\sigma_o^2}, \quad b = \theta_o$$

$$D = A + B, \quad c = \frac{1}{A + B} (Aa + Bb)$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto e^{-\frac{1}{2} \left(\frac{n}{\sigma^2} + \frac{1}{\sigma_o^2} \right) \left(\theta - \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\sigma_o^2}} \left(\frac{n \bar{x}}{\sigma^2} + \frac{\theta_o}{\sigma_o^2} \right) \right)^2}$$

$$\text{Let; } \sigma'^2 = \frac{1}{\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_o^2} \right)}, \quad \theta' = \frac{1}{\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_o^2} \right)} \left(\frac{n \bar{x}}{\sigma^2} + \frac{\theta_o}{\sigma_o^2} \right)$$

$$p(\theta | x_1, x_2, \dots, x_n) \propto e^{-\frac{1}{2 \sigma'^2} (\theta - \theta')^2}$$

$$p(\theta | x_1, x_2, \dots, x_n) \sim N(\theta', \sigma'^2)$$

The complete p.d.f. of the posterior probability is ;

$$p(\theta | x_1, x_2, \dots, x_n) = \frac{1}{\sqrt{2 \pi \sigma'^2}} e^{-\frac{1}{2 \sigma'^2} (\theta - \theta')^2}$$

$$\therefore E(\theta | X) = \hat{\theta}_{\text{Bayes}} = \theta' = \frac{1}{\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_o^2} \right)} \left(\frac{n \bar{X}}{\sigma^2} + \frac{\theta_o}{\sigma_o^2} \right)$$