Statistical Inference

Department of Statistics & Informative

Fourth Stage

First Semester (2023-2024)

Lecturer: ***Zainab A. M***

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Contents of The First Semester

Chapter one

First review all subjects and laws of Mathematical Statistics

Statistical Distributions

First: The Discrete Distributions:

1) Discrete Uniform Distribution X ~ D.U (n)

$$f(x;n)=\begin{cases}\dfrac{1}{n} & x=1,2,\ldots,n\\ 0 & o.w\end{cases}$$

$$mean=\frac{n+1}{2} \quad , \quad \operatorname{var}(X)=\frac{n^2-1}{12}$$

2) Bernoulli Distribution X ~ Ber (θ)

$$f(x;\theta)=p(x)=\begin{cases}\theta^x(1-\theta)^{1-x} & , \quad x=0,1\\ 0 & o.w\end{cases}$$

$$mean=E(X)=\theta \qquad \operatorname{var}(X)=\theta(1-\theta)$$

3) Binomial Distribution X ~ Bin (n, θ)

$$f(x;n,\theta)=\begin{cases}C_x^n\,\theta^x(1-\theta)^{n-x} & , \quad x=0,1,2,\ldots.,n\\ 0 & o.w\end{cases}$$

$$mean=E(X)=n\theta \qquad \operatorname{var}(X)=n\theta(1-\theta)$$

4) Negative Binomial Distribution X ~ N.Bin (r, θ)

$$f(x;r,\theta)=\begin{cases}C_x^{x+r-1}\theta^r(1-\theta)^x & , \quad x=0,1,2,\ldots..\\ 0 & o.w\end{cases}$$

$$mean=E(X)=\frac{r(1-\theta)}{\theta} \quad , \quad v(X)=\frac{r(1-\theta)}{\theta^2}$$

5) Geometric Distribution X ~ Geo (θ)

$$f(x;\theta)=\begin{cases}\theta(1-\theta)^x & , \quad x=0,1,2,\ldots\ldots..\\ 0 & o.w\end{cases}$$

$$mean=E(X)=\frac{(1-\theta)}{\theta} \quad , \quad v(X)=\frac{(1-\theta)}{\theta^2}$$

6) The Poisson Distribution X ~ Poi (θ)

$$f(x;\theta) = \begin{cases} \dfrac{e^{-\theta}\,\theta^{x}}{x!} & , \quad x = 0,1,2,\ldots. \\ 0 & o.w \end{cases}$$

$$mean = E(X) = \theta \qquad , \quad v(X) = \theta$$

Second: The Continuous Distributions:

1. Continuous Uniform Distribution X ~ C.U(a , b)

$$f(x;a,b) = \begin{cases} \dfrac{1}{b-a} & , \quad a \le x \le b \\ 0 & o.w \end{cases}$$

$$mean = E(X) = \frac{a+b}{2} \qquad , \quad v(X) = \frac{(b-a)^2}{12}$$

2. Beta Distribution X ~ *Beta* (α, β)

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{1}{B(\alpha,\beta)}\, x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & o.w \end{cases}$$

$$mean = E(X) = \frac{\alpha}{\alpha+\beta} \qquad , \quad v(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

3. Gamma Distribution:

a) Gamma, Distribution

1. X ~ Γ(α, θ)

$$f(x;\alpha,\theta) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\theta^{\alpha}}\, x^{\alpha-1} e^{-x/\theta} & , \quad x > 0 \\ 0 & o.w \end{cases}$$

$$mean = E(X) = \alpha\theta \qquad , \quad v(X) = \alpha\theta^2$$

2. X ~ Γ(α, 1/θ)

$$f(x;\alpha,\theta) = \begin{cases} \dfrac{\theta^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\theta x} & , \quad x > 0 \\ 0 & o.w \end{cases}$$

$$mean = E(X) = \alpha/\theta \qquad , \quad v(X) = \alpha/\theta^2$$

b) Exponential Distribution

1. X ~ Exp(θ)

$$f(x;\,\theta) = \begin{cases} \dfrac{1}{\theta}\, e^{-x/\theta} & ,\; x > 0 \\ 0 & o.w \end{cases}$$

$$mean = E(X) = \theta \quad , \quad v(X) = \theta^2$$

2. X ~ Exp(1/θ)

$$f(x;\,\theta) = \begin{cases} \theta\, e^{-\theta x} & ,\; x > 0 \\ 0 & o.w \end{cases}$$

$$mean = E(X) = 1/\theta \quad , \quad v(X) = 1/\theta^2$$

c) Chi-Square Distribution X ~ $\chi^2_{(r)}$

$$f(x;r) = \begin{cases} \dfrac{1}{\Gamma(r/2)\, 2^{r/2}}\, x^{\frac{r}{2}-1} e^{-x/2} & ,\; x > 0 \quad ,\; r > 0 \\ 0 & o.w \end{cases}$$

$$mean = E(X) = r \quad , \quad v(X) = 2r$$

4. Normal (Gaussian) Distribution X ~ N(θ, σ^2)

$$f(x;\,\theta,\sigma) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2\sigma^2}(x-\theta)^2} & ,\; -\infty < x < \infty \\ 0 & o.w \end{cases}$$

$$mean = E(X) = \theta \quad , \quad v(X) = \sigma^2 \quad , \; -\infty < \theta < \infty \quad , \quad \sigma > 0$$

Chapter Two

Distributions of Functions of Random Variables

First: Transformations of the Discrete Random Variables

If X is a discrete r.v., having p.d.f. $f(x)$, taking values in sample space $\mathcal{S}$, $A = \{x;\ x = x_1, x_2, \ldots, x_n\}$, at each of which $f(x) > 0$, and let a r.v. $y = g(x)$ define a **one-to-one transformation** that maps A onto B, $B = \{y;\ y = y_1, y_2, \ldots, y_n\}$.
If we solve $y = g(x)$ for x in terms of y, say, $x = w(y)$, then for each $y \in B$, we have $x = w(y) \in A$.
Then to find the p.d.f. of Y, is given as follows;

$$f(y) = p(Y = y) = p(X = w(y)) = \begin{cases} f[w(y)] & , \quad y \in B \\ 0 & o.w \end{cases}$$

Ex: Let $X \sim poi(\theta)$ and $Y = 4X$ by using transformation technique, find the p.d.f. of Y.
Sol:

$$\because X \sim poi(\theta) \quad \therefore p.d.f.\ of\ X = f(x) = \frac{e^{-\theta}\theta^{x}}{x!} \quad , \ x = 0, 1, 2, \ldots\ldots$$

$$A = \{x : x = 0, 1, 2, \ldots\ldots\} \quad , \ f(x) > 0$$

$$\because Y = 4X$$

$$\therefore B = \{y : y = 0, 4, 8, \ldots\ldots\} \quad , \ f(y) > 0$$

$$f(y) = p(Y = y) = p(4X = y) = p(X = \tfrac{y}{4}) = \begin{cases} \dfrac{e^{-\theta}\theta^{\frac{y}{4}}}{\frac{y}{4}!} & , \ y = 0, 4, 8, \ldots\ldots \\ 0 & o.w \end{cases}$$

Ex: Let X have the binomial p.d.f. . $X \sim Bin(3, 2/3)$, where $Y = X^2$, by using one-to-one transformation, find the p.d.f. of Y.
Sol:

$$\because X \sim b(3, 2/3) \quad \Rightarrow \therefore f(x) = \begin{cases} C_x^3 (\frac{2}{3})^x (\frac{1}{3})^{3-x} & x = 0,1,2,3 \\ 0 & o.w \end{cases}$$

$$A = \{x : x \in R_X = 0, 1, 2, 3\} \quad , \ f(x) > 0$$

$$\because Y = X^2 \quad \Rightarrow \quad \therefore B = \{y : y \in R_Y = 0, 1, 4, 9\} \quad , \ f(y) > 0$$

In general, $Y = X^2$ does not define a one-to-one transformation, but here there are not negative values of x in $A = \{x;\ x = 0, 1, 2, 3\}$, then $x = w(y) = \sqrt{y}$ $(not\ -\sqrt{y})$, and so;

$$f(y) = p(Y = y) = p(X^2 = y) = p(X = \mp\sqrt{y}) = p(X = \sqrt{y})$$

$$= \begin{cases} \dfrac{3!}{(\sqrt{y})!\,(3-\sqrt{y})!} (\frac{2}{3})^{\sqrt{y}} (\frac{1}{3})^{3-\sqrt{y}} & y = 0,1,4,9 \\ 0 & o.w \end{cases}$$

Definition for the J.P.D.F.

Let $f(x_1, x_2)$ be the j.p.d.f. of two discrete r.v.'s X_1 and X_2 with A the (two dimensional) set of points. Which $f(x_1, x_2) > 0$, let $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ define a one-to-one transformation that maps A onto B (two dimensional), then the j.p.d.f. of the two new r.v.'s $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$ is given;

$$f(y_1, y_2) = p(Y_1 = y_1, Y_2 = y_2) = \begin{cases} f[w_1(y_1, y_2), w_2(y_1, y_2)] & , (y_1, y_2) \in B \\ 0 & o.w \end{cases}$$

Ex: Let X_1 and X_2 be two stochastically independent r.v.'s that have Poisson distribution with means θ_1, θ_2 respectively, the j.p.d.f. of X_1 and X_2 is;

$$f(x_1, x_2) = \begin{cases} \dfrac{\theta_1^{x_1}\, \theta_2^{x_2}\, e^{-\theta_1 - \theta_2}}{x_1!\, x_2!} & , x_1 = 0,1,2,3,\ldots.. \quad , x_2 = 0,1,2,3,\ldots.. \\ 0 & o.w \end{cases}$$

Where $Y_1 = X_1 + X_2$, $Y_2 = X_2$. **Find:** the j.p.d.f. of Y_1 and Y_2. and $f_1(y_1)$. , **HW**: $f_2(y_2)$

Sol:

$\because X_1 \text{ and } X_2 \sim Poi(\theta_1, \theta_2)$

$A = \{(x_1, x_2) : x_1 = 0,1,2,\ldots.. \quad , x_2 = 0,1,2,\ldots..\} \ , \ f(x_1, x_2) > 0$

$B = \{(y_1, y_2) : y_1 = 0,1,2,\ldots.. \quad , y_2 = 0,1,2,\ldots., y_1\} \ , \ f(y_1, y_2) > 0$

$because \ ; \ y_2 = x_2 \Rightarrow \because x_2 = y_1 - x_1 \Rightarrow \therefore y_2 = y_1 - x_1$

$when \ x_1 = 0 \Rightarrow y_2 = y_1 \ (\max) \ldots (y_1 - 1, y_1 - 2, \ldots.) \ldots \ when \ x_1 = \infty \Rightarrow y_2 = \infty - \infty = 0 \ (\min)$

$\because y_1 = x_1 + x_2 \Rightarrow x_1 = y_1 - x_2 \Rightarrow x_1 = y_1 - y_2$

$\quad y_2 = x_2 \qquad\qquad\qquad \Rightarrow x_2 = y_2$

$\therefore the\ j.p.d.f.\ of\ Y_1\ and\ Y_2\ is;$

$$f(y_1, y_2) = p(Y_1 = y_1, Y_2 = y_2) = p(X_1 = y_1 - y_2, X_2 = y_2)$$

$$= \begin{cases} \dfrac{\theta_1^{y_1 - y_2}\, \theta_2^{y_2}\, e^{-\theta_1 - \theta_2}}{(y_1 - y_2)!\, y_2!} & , (y_1, y_2) \in B \\ 0 & o.w \end{cases}$$

The marginal p.d.f. of Y_1 is given by;

$$f_1(y_1) = \sum_{y_2=0}^{y_1} f(y_1, y_2) = e^{-\theta_1 - \theta_2} \sum_{y_2=0}^{y_1} \frac{\theta_1^{y_1 - y_2}\, \theta_2^{y_2}}{(y_1 - y_2)!\, y_2!} \} \times \frac{y_1!}{y_1!}$$

$$= \frac{e^{-\theta_1 - \theta_2}}{y_1!} \sum_{y_2=0}^{y_1} \frac{y_1!}{(y_1 - y_2)!\, y_2!} \theta_1^{y_1 - y_2}\, \theta_2^{y_2} = \frac{e^{-\theta_1 - \theta_2}}{y_1!} \sum_{y_2=0}^{y_1} C_{y_2}^{y_1}\, \theta_1^{y_1 - y_2}\, \theta_2^{y_2}$$

$$= \frac{e^{-\theta_1 - \theta_2} (\theta_1 + \theta_2)^{y_1}}{y_1!} \quad , \quad y_1 = 0,1,2,\ldots...$$

That is, $Y_1 = X_1 + X_2$ has a Poisson distribution with parameter $(\theta_1 + \theta_2)$.

Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ denote a random sample and be independent identically distributed r.v's with a p.d.f. $f(x)$, and let $Y_1 < Y_2 < \ldots < Y_n$ be their ascending ordered values, i.e.;

Y_1: is a smallest value of $(X_1, X_2, \ldots, X_n)$ (min).
Y_2: is the second smallest value of $(X_1, X_2, \ldots, X_n)$.
.
.
Y_n: the largest value of $(X_1, X_2, \ldots, X_n)$ (max).

Then Y_i $(i = 1, 2, \ldots, n)$ is called the i-th order statistic of the random sample $X_1, X_2, \ldots, X_n$. and $Y_1 < Y_2 < \ldots < Y_n$ are called the order statistics corresponding of the random sample $X_1, X_2, \ldots, X_n$.

Then the j.p.d.f. of $X_1, X_2, \ldots, X_n$ is given by;

$$f(x_1, x_2, \ldots, x_n) = f(x_1).\, f(x_2) \ldots f(x_n)$$

The j.p.d.f. of the order statistics $Y_1, Y_2, \ldots, Y_n$ is given by;

$$g(y_1, y_2, \ldots, y_n) = (n!)\; g(y_1)\; g(y_2) \ldots g(y_n)$$

$$= \begin{cases} (n!) \prod_{i=1}^{n} g(y_i) & , \quad a < y_1 < y_2 < \ldots < y_n < b \\ 0 & o.w \end{cases}$$

Explain:

Let $(n = 2)$, then we have two probabilities;

$X_1 > X_2$	or	$X_1 < X_2$
$Y_1 = X_2$		$Y_1 = X_1$
$Y_2 = X_1$		$Y_2 = X_2$

Discrete

$$g(y_1, y_2) = g(y_1 = x_2)\; g(y_2 = x_1) + g(y_1 = x_1)\; g(y_2 = x_2)$$

$$= (2!)\; g(y_1)\; g(y_2)$$

$$= \begin{cases} (2!) \prod_{i=1}^{2} g(y_i) & , \quad a < y_1 < y_2 < b \\ 0 & o.w \end{cases}$$

When $(n = 3)$

$$g(y_1, y_2, y_3) = (3!)\; g(y_1)\; g(y_2)\; g(y_3)$$

$$= \begin{cases} (3!) \prod_{i=1}^{3} g(y_i) & , \quad a < y_1 < y_2 < y_3 < b \\ 0 & o.w \end{cases}$$

Continuous

When $(n = 2)$

$$J_1 = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \quad , \quad J_2 = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

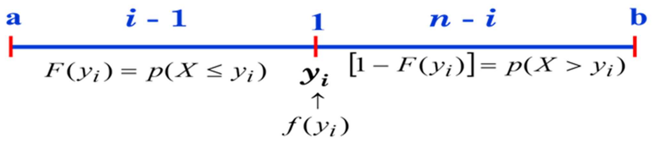
Note: Always J equal to one because be continuous

$$g(y_1, y_2) = g(y_1 = x_2)\ g(y_2 = x_1)\ J_1 + g(y_1 = x_1)\ g(y_2 = x_2)\ J_2$$

The Marginal P.D.F. of an Individual Order Statistics

The marginal p.d.f. of the *i*-th order statistics is given by:

$$g(y_i) = \frac{n!}{(i-1)!(n-i)!} f(y_i)[F(y_i)]^{i-1}[1-F(y_i)]^{n-i} \quad , \quad a < y_i < b$$



The P.D.F. of the Smallest Order Statistics

If (i = 1) then the distribution of y_1 is given by:

$$g(y_1) = \frac{n!}{(1-1)!(n-1)!} f(y_1)[F(y_1)]^{1-1}[1-F(y_1)]^{n-1} \quad , \quad a < y_1 < b$$

$$g(y_1) = \frac{n(n-1)!}{(n-1)!} f(y_1)\,[1-F(y_1)]^{n-1}$$

$$g(y_1) = n\, f(y_1)\,[1-F(y_1)]^{n-1} \quad , \quad a < y_1 < b$$

The P.D.F. of the Largest Order Statistics

If ($i = n$) then the distribution of y_n is given by:

$$g(y_n) = \frac{n!}{(n-1)!(n-n)!} f(y_n)[F(y_n)]^{n-1}[1-F(y_n)]^{n-n} \quad , \quad a < y_n < b$$

$$g(y_n) = \frac{n(n-1)!}{(n-1)!} f(y_n)\ [F(y_n)]^{n-1}$$

$$g(y_n) = n\, f(y_n)\,[F(y_n)]^{n-1} \quad , \quad a < y_n < b$$

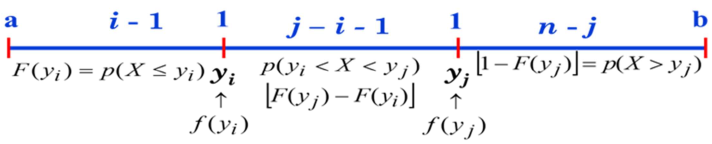
The Joint Probability Density Fun. of Two Order Statistics

The joint p.d.f. of any two order statistics Y_i and Y_j ($i < j$) is given by:

$$g(y_i, y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times$$

$$\times\ [F(y_i)]^{i-1}[F(y_j) - F(y_i)]^{j-i-1}[1-F(y_j)]^{n-j} f(y_i)\, f(y_j) \quad , \quad a < y_i < y_j < b$$

$$0 \qquad\qquad o.w$$



Ex: let $X_1, X_2, \ldots, X_n$ be a random sample of size (n) rssn taken from C.U(0,1). let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of this sample. **Find** the p.d.f. of Y_1 and Y_n, the j.p.d.f. of Y_1 and Y_n

Sol.: $X \sim C.U(0,1)$

$$\therefore f(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & o.w \end{cases} = \begin{cases} 1 & , 0 \le x \le 1 \\ 0 & o.w \end{cases}$$

$$F(x) = p(X \le x) = \int_0^x 1\,dx = \begin{cases} 0 & , x \le 0 \\ x & , 0 < x < 1 \\ 1 & , x \ge 1 \end{cases}$$

$$g(y_1) = n\ f(y_1)\,[1 - F(y_1)]^{n-1} \quad , \quad a \le y_1 \le b$$

$$when\ \ x = y_1 \ \Rightarrow\ \therefore\ f(y_1) = 1 \ ,\ F(y_1) = y_1$$

$$\therefore g(y_1) = n\,(1)\,[1 - y_1]^{n-1} \quad , \quad 0 \le y_1 \le 1$$

$$= \begin{cases} n(1-y_1)^{n-1} & , 0 \le y_1 \le 1 \\ 0 & o.w \end{cases}$$

$$g(y_n) = n\ f(y_n)\,[F(y_n)]^{n-1} \quad , \quad a \le y_n \le b$$

$$when\ \ x = y_n \ \Rightarrow\ \therefore\ f(y_n) = 1$$

$$\therefore\ g(y_n) = n\,(1)\,[y_n]^{n-1} \quad , \quad 0 \le y_n \le 1$$

$$= \begin{cases} n\,y_n^{n-1} & , 0 \le y_n \le 1 \\ 0 & o.w \end{cases}$$

$$g(y_i, y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times$$

$$\times\ [F(y_i)]^{i-1}\,[F(y_j) - F(y_i)]^{j-i-1}\,[1 - F(y_j)]^{n-j} f(y_i)\,f(y_j) \quad , \quad a < y_i < y_j < b$$

$$0 \qquad o.w$$

When, $i = 1\ ,\ j = n$

$$\therefore g(y_1, y_n) = \frac{n!}{0!(n-2)!0!}(y_n - y_1)^{n-2} = \frac{n(n-1)(n-2)!}{(n-2)!}(y_n - y_1)^{n-2}$$

$$= n(n-1)\,(y_n - y_1)^{n-2} \quad , \ 0 < y_1 < y_n < 1$$

Chapter Three

Statistical Inference

Statistical Inference: making conclusions about the whole population on the basis of a sample, i.e., use a random sample to learn something about a large population.
Precondition for statistical inference: A sample is randomly selected from the population.

Concepts and Important Definitions about Stat. Inference

1. $\underline{X} = (X_1, X_2, \ldots, X_n) \equiv$ rssn $\equiv$ Data
2. Statistic: is a function of the random variable (r.v.) only in the sample data.
3. Parameter: It is a characteristic or a measure that is calculated from the population under study. **Ex:** The unemployment rate in Erbil. The average of assumption life for a particular device. **[Parameter = Statistic ± It's Error]**.
4. Population parameters are denoted using Greek letters μ (mean), σ (standard deviation), π (proportion). Sample values are denoted $\overline{x}$ (mean), S (standard deviation), p (proportion).

5. Estimator: is a function.
6. Estimate: is a value of the estimator.

$$\overline{X} = \underbrace{\frac{\sum X_i}{n}}_{Estimator} = \underbrace{15}_{Estimate}$$

7.

$$Quantitative\ Variable \Rightarrow Standard\ \ Error = SE(Mean) = S / \sqrt{n}$$

$$Qualitative\ Variable \Rightarrow Standard\ \ Error = SE(p) = \sqrt{p(1-p)/n}$$

There are two steps to make inference:

1. Estimation of the population parameters

a) Point Estimation.
b) Intervals Estimation.

2. Testing of Hypotheses about the right values of population parameters.

Estimation of Parameters

First: Point Estimation

Let $X_1, X_2, \ldots, X_n$ be a rssn from the p.d.f. $f(x;\theta)$, θ is unknown. We want to estimate θ from the information in the data.

$$\hat{\theta} = estimator\ \ of\ \ \theta$$

Properties of Estimator:

1. Unbiased Estimator

An estimator $\left(\hat{\theta} = t(x_1, \ldots, x_n)\right)$ from a sample of size (n) with p.d.f. $f(x;\theta)$ is said to be an unbiased estimator for a population parameter θ if:

$$E(\hat{\theta}) = \theta$$

The quantity $\left(E(\hat{\theta}) - \theta\right)$ is called bias of an estimator $\hat{\theta} = t(X)$ of θ.

$$bias\ (\hat{\theta}) = E(\hat{\theta}) - \theta$$

Ex: In a random sample of size (n) taken from exponential dist^n Exp(θ). Show that;

1. $T_1 = \overline{X}$ is unbiased estimator for the parameter (θ).

2. $T_2 = \frac{n}{n+1} \overline{X}^2$ is unbiased estimator for the parameter (θ^2).

Sol: 1)

$$E(T_1) \overset{?}{=} \theta$$

$$E(T_1) = E(\overline{X}) = \frac{1}{n} E(\sum X_i) = \frac{1}{n}\, n\, E(X) = \theta$$

$\therefore\ \overline{X}$ is unbiased estimator for θ.

2)

$$E(T_2) = \frac{n}{n+1} E(\overline{X})^2$$

$$= \frac{n}{n+1}\left(V(\overline{X}) + (E(\overline{X}))^2\right) = \frac{n}{n+1}\left(\frac{V(X)}{n} + \theta^2\right)$$

$$= \frac{n}{n+1}\left(\frac{\theta^2}{n} + \theta^2\right) = \frac{n}{n+1}\left(\frac{\theta^2 + n\theta^2}{n}\right)$$

$$= \frac{n}{n+1}\left(\frac{\theta^2(n+1)}{n}\right) = \theta^2$$

$\therefore\ T_2 = \frac{n}{n+1} \overline{X}^2$ is unbiased estimator for θ^2.

Ex: In a random sample of size (n) from normal dist^n N(θ, σ^2). Show that;

1) $S^2 = \frac{1}{n-1} \sum (X_i - \overline{X})^2$ is unbiased estimator for the parameter (σ^2).

2) Is $T = \overline{X}^2$ unbiased estimator for θ^2.

Sol: 1)

$$X \sim N(\theta, \sigma^2)$$

$$E(\overline{X}) = E\left(\frac{\sum X_i}{n}\right) = \frac{n}{n} E(X) = \theta$$

$$V(\overline{X}) = V\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} \sum V(X_i) = \frac{n}{n^2} \sigma^2 = \frac{\sigma^2}{n}$$

$$\overline{X} \sim N(\theta, \frac{\sigma^2}{n})$$

$$E(S^2) = \frac{1}{n-1} E\left(\sum (X_i - \overline{X})^2\right) = \frac{1}{n-1} E\left(\sum X_i^2 - n\overline{X}^2\right)$$

$$= \frac{n}{n-1}\left(E(X^2) - E(\overline{X})^2\right)$$

$$= \frac{n}{n-1}\left(\left(V(X) + (E(X))^2\right) - \left(V(\overline{X}) + (E(\overline{X}))^2\right)\right)$$

$$= \frac{n}{n-1}\left(\left(\sigma^2 + \theta^2\right) - \left(\frac{\sigma^2}{n} + \theta^2\right)\right) = \frac{n}{n-1}\left(\sigma^2 + \theta^2 - \frac{\sigma^2}{n} - \theta^2\right)$$

$$= \frac{n}{n-1}\left(\sigma^2 - \frac{\sigma^2}{n}\right) = \frac{n}{n-1}\left(\frac{n\sigma^2 - \sigma^2}{n}\right) = \frac{n}{n-1}\,\frac{\sigma^2(n-1)}{n} = \sigma^2$$

$$\therefore\ S^2 = \frac{1}{n-1}\sum (X_i - \overline{X})^2 \ \textit{is unbiased estimator for } \sigma^2$$

2)

$$E(\overline{X})^2 \overset{?}{=} \theta^2$$

$$V(\overline{X}) = E(\overline{X})^2 - (E(\overline{X}))^2$$

$$E(\overline{X})^2 = V(\overline{X}) + (E(\overline{X}))^2 = \frac{\sigma^2}{n} + \theta^2 \neq \theta^2$$

$$\therefore\ \hat{\theta} = \overline{X}^2 \ \textit{is not unbiased estimator for } \theta^2.$$

Then; what is to be unbiased estimator for θ^2.

Now from both sides we subtract $\frac{\sigma^2}{n}$;

$$E(\overline{X})^2 - \frac{\sigma^2}{n} = \frac{\sigma^2}{n} + \theta^2 - \frac{\sigma^2}{n} = \theta^2$$

$$\therefore\ \hat{\theta} = \overline{X}^2 - \frac{\sigma^2}{n} \ \textit{is unbiased estimator for } \theta^2$$

Unbiased in Limit

An estimator $\hat{\theta}$ for known parameter θ of p.d.f. $f(x;\theta)$ is unbiased in limit if:

$$\lim_{n \to \infty} E(\hat{\theta}) = \theta$$

Ex: In a rss(n) from uniform dist^n C.U(0, θ).

1) Is Y_n unbiased in limit estimator for θ; (Note: Y_n estimator θ).

2) Is $\overline{X}$ unbiased in limit estimator for θ.

3) Is $\overline{X}$ unbiased in limit estimator for $\theta/2$.

Sol: 1)

$$f(x) = \frac{1}{b-a} = \frac{1}{\theta - 0} = \frac{1}{\theta} \quad , 0 < x < \theta$$

$$F(y_i) = p(X \le y_i) = \int_0^{y_i} \frac{1}{\theta}\,dx = \frac{y_i}{\theta}$$

$$g(y_n) = n\ f(y_n)\,(F(y_n))^{n-1} = n\,\frac{1}{\theta}\left(\frac{y_n}{\theta}\right)^{n-1} = \frac{n\,y_n^{n-1}}{\theta^n}\quad ,0 < y_n < \theta$$

$$E(Y_n) = \int_{R_{y_n}} y_n\ g(y_n)\ dy_n = \int_0^\theta y_n\ \frac{n\,y_n^{n-1}}{\theta^n}\ dy_n = \frac{n}{\theta^n}\int_0^\theta y_n^n\ dy_n$$

$$= \frac{n}{\theta^n}\ \frac{y_n^{n+1}}{n+1}\Bigg|_0^\theta = \frac{n}{\theta^n}\ \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\ \theta \neq \theta\ \rightarrow \therefore Y_n \text{ is not unbiased est. for } \theta$$

$$\lim_{n\to\infty} E(Y_n) = \lim_{n\to\infty} \frac{n}{n+1}\ \theta = (1)\ \theta = \theta\ \rightarrow \therefore Y_n \text{ is unbiased in limit est. for } \theta$$

2. Consistency Estimator

Definition: An estimator $\hat{\theta}$ of the parameter θ of $f(x;\theta)$ is called consistent estimator for θ if;

$$\lim_{n\to\infty} p\left(\left|\hat{\theta} - \theta\right| < \varepsilon\right) = 1 \quad , \quad \forall\ \varepsilon > 0$$

$$or\ ;\ \lim_{n\to\infty} p\left(\left|\hat{\theta} - \theta\right| \geq \varepsilon\right) = 0$$

Note: Consistency means the estimator equal to the parameter or converges stochastically to the parameter θ.

A consistent estimator: That the estimator gets closer to the parameter value as n increases without limit.

$$|\,\hat{\theta} - \theta\,| \ \Rightarrow \text{called estimated error}$$

$$\left.\begin{aligned} p\left(\left|\hat{\theta} - \theta\right| < \varepsilon\right) &\geq 1 - \frac{v(\hat{\theta})}{\varepsilon^2} \\ p\left(\left|\hat{\theta} - \theta\right| \geq \varepsilon\right) &< \frac{v(\hat{\theta})}{\varepsilon^2} \end{aligned}\right\} \rightarrow (\text{Chebycheve inequality})$$

Theorem: Let $\hat{\theta}$ be an estimator for the population parameter θ of $f(x;\theta)$, then $\hat{\theta}$ is said to be consistent estimator for θ if:

$$1)\ \lim_{n\to\infty} E(\hat{\theta}) = \theta \qquad\qquad 2)\ \lim_{n\to\infty} V(\hat{\theta}) = 0$$

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from Poisson dist[n], show that $\hat{\theta} = \overline{X}$ is consistent estimator for θ.

Sol:

First Method ;

$$1)\quad p\left(\left|\hat{\theta}-\theta\right|<\varepsilon\right)\ge 1-\frac{v(\hat{\theta})}{\varepsilon^{2}}$$

$$\lim_{n\to\infty} p\left(\left|\hat{\theta}-\theta\right|<\varepsilon\right)\ge \lim_{n\to\infty}\left(1-\frac{v(\hat{\theta})}{\varepsilon^{2}}\right)$$

$$\hat{\theta}=\overline{X}\ ,\ v(\hat{\theta})=v(\overline{X})=\frac{v(X)}{n}=\frac{\theta}{n}$$

$$\lim_{n\to\infty} p\left(\left|\overline{X}-\theta\right|<\varepsilon\right)\ge \lim_{n\to\infty}\left(1-\frac{\theta}{n\,\varepsilon^{2}}\right)$$

$$\lim_{n\to\infty} p\left(\left|\overline{X}-\theta\right|<\varepsilon\right)=1$$

$$2)\quad p\left(\left|\hat{\theta}-\theta\right|\ge\varepsilon\right)<\frac{v(\hat{\theta})}{\varepsilon^{2}}$$

$$\lim_{n\to\infty} p\left(\left|\hat{\theta}-\theta\right|\ge\varepsilon\right)<\lim_{n\to\infty}\frac{\theta}{n\,\varepsilon^{2}}$$

$$\lim_{n\to\infty} p\left(\left|\overline{X}-\theta\right|\ge\varepsilon\right)=0$$

Second Method ;

$$1)\quad \lim_{n\to\infty} E(\hat{\theta})=\lim_{n\to\infty} E(\overline{X})=\theta$$

$$2)\quad \lim_{n\to\infty} v(\hat{\theta})=\lim_{n\to\infty}\frac{\theta}{n}=0$$

$\therefore\ \hat{\theta}=\overline{X}$ *is consistent estimator for* θ.

The Score Function

The score function is the partial derivative of Log the function $f(x;\theta)$ with respect to the parameter θ, is defined as;

$$S(x;\theta)=\frac{\partial}{\partial\theta}\ln f(x;\theta)=\frac{1}{f(x;\theta)}\frac{\partial}{\partial\theta}f(x;\theta)$$

Properties

1) The mean of the score is zero, E(S(X; θ)) = zero

Proof:

$$E\big(S(X;\theta)\big)=\int_{R_X} s(x;\theta)\,f(x;\theta)\,dx=\int_{R_X}\frac{1}{f(x;\theta)}\frac{\partial}{\partial\theta}f(x;\theta)\,f(x;\theta)\,dx$$

$$=\int_{R_X}\frac{\partial}{\partial\theta}f(x;\theta)\,dx=\frac{\partial}{\partial\theta}\int_{R_X}f(x;\theta)\,dx=\frac{\partial}{\partial\theta}(1)=zero$$

2) The variance of the score is known as the Fisher Information (F.I), which is measure the information in the sample $\mathcal{S}$ about the parameter *θ*, and can be written as;

$$F.I = I(\theta) = E\left(\frac{\partial}{\partial\theta}\ln f(x;\theta)\right)^2 \qquad , \text{because mean} = \text{zero}$$

Or;

$$F.I = I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right)$$

If Fisher Information multiply by (n), *we get;*

$$n\,I(\theta) = F.I \text{ in a } rss(n)$$

Ex: Let $X_1, \ldots, X_n$ be a rss$n$ from exponential dist^n Exp($1/\theta$). Find the F.I. of X.

Sol:

$$f(x;\theta) = \theta e^{-\theta x} \qquad , x > 0$$

$$\ln f(x;\theta) = \ln(\theta) - \theta x$$

$$\frac{\partial}{\partial\theta}\ln f(x;\theta) = \frac{1}{\theta} - x$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) = -\frac{1}{\theta^2}$$

$$F.I = -E\left(\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right) = \frac{1}{\theta^2} \qquad \Rightarrow \qquad \therefore n\,I(\theta) = F.I. \text{ in a } rss(n) = \frac{n}{\theta^2}$$

3. Sufficiency Estimator

Sufficiency estimator is containing all the information in the data about the parameter *θ*.

First Method (Fisher Information)

Definition 1: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from the dist^n with p,d.f. $f(x\,;\,\theta)$, an estimator $\hat{\theta}$ is sufficient estimator for the parameter *θ* if the Fisher information in $\hat{\theta}$ is equal to the Fisher information in a rss(n).

Ex: Show that $\overline{X}$ is sufficient estimator for the mean of $N(\theta, \sigma^2)$.

Sol:

$$F.I \text{ in a } rssn = F.I \text{ in } \hat{\theta}$$

$$F.I \text{ in a } rssn = -n\,E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right)$$

$$f(x;\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}$$

$$\ln f(x;\theta,\sigma^2) = \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}(x-\theta)^2$$

$$\frac{\partial \ln f(x;\theta,\sigma^2)}{\partial \theta} = zero - \frac{2(x_i-\theta)(-1)}{2\sigma^2} = \frac{(x_i-\theta)}{\sigma^2}$$

$$\frac{\partial^2 \ln f(x;\theta,\sigma^2)}{\partial \theta^2} = -\frac{1}{\sigma^2}$$

$$-nE\left(\frac{\partial^2 \ln f(x;\theta,\sigma^2)}{\partial \theta^2}\right) = \frac{n}{\sigma^2} \quad is\ F.I.\ in\ a\ rssn$$

$$X \sim N(\theta, \sigma^2)$$

$$\overline{X} \sim N(\theta, \frac{\sigma^2}{n})$$

$$g(\overline{x};\theta,\frac{\sigma^2}{n}) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\overline{x}-\theta)^2}$$

$$\ln\left(g(\overline{x};\theta,\frac{\sigma^2}{n})\right) = \ln\left(\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\right) - \frac{n}{2\sigma^2}(\overline{x}-\theta)^2$$

$$\frac{\partial \ln g(\overline{x};\theta,\frac{\sigma^2}{n})}{\partial \theta} = zero - \frac{2n(\overline{x}-\theta)(-1)}{2\sigma^2} = \frac{n(\overline{x}-\theta)}{\sigma^2}$$

$$\frac{\partial^2 \ln g(\overline{x};\theta,\frac{\sigma^2}{n})}{\partial \theta^2} = -\frac{n}{\sigma^2}$$

$$-E\left(\frac{\partial^2 \ln g(\overline{x};\theta,\frac{\sigma^2}{n})}{\partial \theta^2}\right) = -E\left(\frac{-n}{\sigma^2}\right) = \frac{n}{\sigma^2} \quad is\ F.I.\ (\hat{\theta} = \overline{x})$$

$$\because F.I\ in\ a\ rssn = F.I\ in\ (\hat{\theta} = \overline{x})$$

$$\therefore \hat{\theta} = \overline{x}\ is\ suff\ est\ for\ \theta$$

Second Method (Conditional)

Definition 2: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from the dist^n with p,d.f. $f(x\ ;\ \theta)$, and $\hat{\theta}$ be an estimator for θ, an estimator $\hat{\theta}$ is sufficient estimator for the parameter θ if the conditional p.d.f. of $(X_1, X_2, \ldots, X_n)$ given $\hat{\theta}$ does not contain the parameter θ:

$$f(x_1, x_2, \ldots, x_n \mid \hat{\theta}) = \frac{f(x_1, x_2, \ldots, x_n; \theta)}{g(\hat{\theta})}$$

Note: If the range depends on the parameter, in this case we can't find F.I; therefore, we use the second method (Conditional).

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from a dist^n with p.d.f.:

$$f(x;\theta) = e^{2\theta - x} \quad , \quad x \ge 2\theta$$

Show that Y_1 is sufficient estimator for the parameter θ.

Sol:

$$f(x_1, x_2, ..., x_n \mid \hat{\theta} = y_1) = \frac{f(x_1, x_2, ..., x_n ; \theta)}{g(y_1)}$$

$\because\ f(x;\theta) = e^{2\theta - x}$, Xs are independent

$$f(x_1, x_2, ..., x_n ; \theta) = f(x_1;\theta) \cdot f(x_2;\theta) \cdots f(x_n;\theta)$$

$$= e^{2\theta - x_1} \times e^{2\theta - x_2} \times ..... \times e^{2\theta - x_n}$$

$$= e^{\sum(2\theta - x_i)}$$

$$= e^{2n\theta - \sum x_i}$$

$$g(y_1) = n\, f(y_1)(1 - F(y_1))^{n-1}$$

$$f(y_1) = e^{2\theta - y_1}$$

$$F(y_1) = p(X \le y_1) = \int_{2\theta}^{y_1} e^{2\theta - x}\, dx = e^{2\theta} \int_{2\theta}^{y_1} e^{-x}\, dx$$

$$= e^{2\theta} \left(-e^{-x}\right)_{2\theta}^{y_1} = e^{2\theta}\,(e^{-2\theta} - e^{-y_1})$$

$$= 1 - e^{2\theta - y_1}$$

$$\therefore\ g(y_1) = n\left(e^{2\theta - y_1}\right)\left(1 - (1 - e^{2\theta - y_1})\right)^{n-1}$$

$$= n\left(e^{2\theta - y_1}\right)\left(e^{2\theta - y_1}\right)^{n-1}$$

$$= n\, e^{2n\theta - n y_1} \qquad , y_1 \ge 2\theta$$

$$\therefore\ f(x_1, x_2, ..., x_n \mid \hat{\theta} = Y_1) = \frac{e^{2n\theta - \sum x_i}}{n\, e^{2n\theta - n y_1}}$$

$$= \frac{e^{-\sum x_i}}{n\, e^{-n y_1}} = \frac{1}{n} e^{-\sum x_i + n y_1} \quad \text{does not contain } \theta$$

$\therefore\ Y_1$ is suff est for θ

Third Method: Factorization Theorem

Definition 3: Let $\hat{\theta}$ be an estimator for the parameter of $f(x;\theta)$ such that the range does not depend on θ. Then the necessary and sufficient condition for an estimator $\hat{\theta}$ to be sufficient estimator, if there are two non-negative functions, such that:

$$f(x_1, x_2, ..., x_n ; \theta) = g(\hat{\theta}; \theta) \cdot H(x)$$

Theorem:

Let $\hat{\theta}$ be sufficient estimator for the parameter θ, and $u(\hat{\theta})$ be a one-to-one transformation, then $u(\hat{\theta})$ is sufficient estimator for θ.

Note: 1) $\bar{x}$ is one to one transformation to $\sum x_i$. $\Rightarrow\ \sum X_i = n\bar{X}$.

2) If we have more than one parameter, we use factorization theorem (third method) for sufficiency.

Ex: Let $X_1, X_2, \ldots, X_n$ be a rssn from Bernoulli dist^n Ber(θ). Show that $\hat{\theta} = \sum X_i$ is sufficient estimator for the parameter θ.

Sol:

$$\because X \sim Ber(\theta)$$

$$f(x;\theta) = \theta^x (1-\theta)^{1-x} \quad , x = 0,1$$

$$f(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i;\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \quad \} \times \frac{C^n_{\sum x_i}}{C^n_{\sum x_i}}$$

$$= C^n_{\sum x_i}\, \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \times \frac{1}{C^n_{\sum x_i}} \quad , \text{ free of } \theta$$

$$= g(\hat{\theta} = \sum x_i; \theta) \times H(x) \qquad \Rightarrow \therefore \hat{\theta} = \sum x_i \text{ is suff est for } \theta$$

Ex: Let $X_1, X_2, \ldots, X_n$ be a rssn from Poisson dist^n Poi(θ), show that $\hat{\theta} = \sum X_i$ is sufficient estimator for θ?

Sol:

$$X \sim Poi(\theta) \quad \Rightarrow f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!} \quad , \because Xs \text{ are independent}$$

$$f(x_1, x_2, \ldots, x_n; \theta) = f(x_1;\theta)\cdot f(x_2;\theta)\cdots f(x_n;\theta)$$

$$= \frac{e^{-\theta}\theta^{x_1}}{x_1!} \times \frac{e^{-\theta}\theta^{x_2}}{x_2!} \times \ldots\ldots \times \frac{e^{-\theta}\theta^{x_n}}{x_n!}$$

$$= \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod (x_i)!} \qquad \text{joint p.d.f.}$$

$$= e^{-n\theta}\theta^{\sum x_i} \times \frac{1}{\prod (x_i)!}$$

$$= g(\hat{\theta} = \sum x_i; \theta) \times H(x)$$

$$\therefore \hat{\theta} = \sum x_i \text{ is suff est for } \theta.$$

Ex: from Exp($1/\theta$). Is $\sum_{i=1}^{n} X_i$ sufficient estimator for θ? (by factorization theorem).

Sol:

$$f(x_1, x_2, \ldots, x_n; \theta) = g(\hat{\theta}; \theta)\cdot H(x)$$

$$X \sim Exp(1/\theta) \quad \Rightarrow f(x;\theta) = \theta e^{-\theta x} \quad , \because Xs \text{ are independent}$$

$$f(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i;\theta) = f(x_1;\theta)\cdot f(x_2;\theta)\cdots f(x_n;\theta)$$

$$= \theta e^{-\theta x_1} \times \theta e^{-\theta x_2} \times \ldots\ldots \times \theta e^{-\theta x_n}$$

$$= \theta^n e^{-\theta \sum x_i} \times 1$$

$$= g(\hat{\theta} = \sum x_i; \theta) \cdot H(x)$$

$$\therefore \hat{\theta} = \sum X_i \text{ is suff est for } \theta.$$

Multi-Parameters Case (Joint Sufficient Estimator)

Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from a (k) parameters dist^n $f(x;\theta_1,\theta_2,\ldots,\theta_k)$, then $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$ are jointly sufficient estimators for parameters $(\theta_1,\theta_2,\ldots,\theta_k)$ respectively if the j.p.d.f. of $(X_1, X_2, \ldots, X_n)$ can be expressed as:

$$f(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_k) = g(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k; \theta_1, \theta_2, \ldots, \theta_k) \cdot H(x)$$

Where; H(x) independent of the parameters $(\theta_1,\theta_2,\ldots,\theta_k)$.

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from Gamma dist^n $\Gamma(\alpha, 1/\theta)$, find the jointly sufficient estimators for the parameters (α, θ).

Sol:

$$f(x;\alpha,\theta) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x}$$

$$f(x_1, x_2, \ldots, x_n; \alpha, \theta) = \left(\frac{\theta^\alpha}{\Gamma(\alpha)}\right)^n \left(\prod_{i=1}^n x_i\right)^{\alpha-1} e^{-\theta \sum x_i}$$

$$= \left(\frac{\theta^\alpha}{\Gamma(\alpha)}\right)^n \left(\prod_{i=1}^n x_i\right)^{\alpha} e^{-\theta \sum x_i} \times \frac{1}{\prod_{i=1}^n x_i}$$

$$= g(\hat{\alpha} = \prod_{i=1}^n x_i, \hat{\theta} = \sum x_i; \alpha, \theta) \cdot H(x)$$

$$\therefore \hat{\alpha} = \prod_{i=1}^n X_i \text{ and } \hat{\theta} = \sum X_i \text{ are jointly suff est for } \alpha \text{ and } \theta.$$

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from normal dist^n $N(\theta, \sigma^2)$, show that $\sum X_i$, $\sum X_i^2$ are the jointly sufficient estimators for the parameters (θ, σ^2) respectively.

Sol: $$f(x;\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \theta)^2}$$

$$f(x_1, x_2, \ldots, x_n; \theta, \sigma^2) = \prod_{i=1}^n f(x_i;\theta) = \left(\sqrt{2\pi\sigma^2}\right)^{-n} e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}$$

$$= \left(\sqrt{2\pi\sigma^2}\right)^{-n} e^{-\frac{1}{2\sigma^2}\sum x_i^2} e^{-\frac{\theta}{\sigma^2}\sum x_i} e^{-\frac{n\theta^2}{2\sigma^2}}$$

$$= g(\hat{\theta} = \sum x_i, \hat{\sigma}^2 = \sum x_i^2; \theta, \sigma^2) \cdot H(x)$$

$$\therefore \sum X_i \text{ and } \sum X_i^2 \text{ are jointly suff est for } \theta \text{ and } \sigma^2.$$

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from C.U$(\theta_1 - \theta_2, \theta_1 + \theta_2)$, and $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics, show that Y_1 and Y_n are the jointly sufficient estimators for the parameters (θ_1, θ_2) respectively.

Sol:

$$f(x;a,b)=\frac{1}{b-a}$$

$$f(x;\theta_1,\theta_2)=\frac{1}{\theta_1+\theta_2-(\theta_1-\theta_2)}=\frac{1}{2\theta_2} \quad , \ \theta_1-\theta_2<x<\theta_1+\theta_2$$

$$f(y_1)=\frac{1}{2\theta_2}$$

$$F(y_1)=p(Y_1\le y_1)=\int_{\theta_1-\theta_2}^{y_1}\frac{1}{2\theta_2}\,dy_1=\frac{1}{2\theta_2}\,y_1\Big|_{\theta_1-\theta_2}^{y_1}=\frac{1}{2\theta_2}\left(y_1-(\theta_1-\theta_2)\right)$$

$$g(y_i,y_j)=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}\times$$
$$\times\ [F(y_i)]^{i-1}\left[F(y_j)-F(y_i)\right]^{j-i-1}\left[1-F(y_j)\right]^{n-j} f(y_i)\,f(y_j) \quad , \quad a<y_i<y_j<b$$
$$0 \qquad\qquad o.w$$

$$g(y_1,y_n)=n\,(n-1)\left[F(y_j)-F(y_i)\right]^{n-2} f(y_1)\,f(y_n) \quad , \ \theta_1-\theta_2<y_1<y_n<\theta_1+\theta_2$$
$$=n\,(n-1)\left(\frac{1}{2\theta_2}(y_n-(\theta_1-\theta_2))-\frac{1}{2\theta_2}(y_1-(\theta_1-\theta_2))\right)^{n-2}\frac{1}{2\theta_2}\times\frac{1}{2\theta_2}$$
$$=n\,(n-1)\frac{1}{(2\theta_2)^n}\,(y_n-y_1)^{n-2} \quad , \ \theta_1-\theta_2<y_1<y_n<\theta_1+\theta_2$$

$$f(x_1,x_2,\dots,x_n;\theta_1,\theta_2)=\prod_{i=1}^{n} f(x_i;\theta_1,\theta_2)=\left(\frac{1}{2\theta_2}\right)^n$$

$$f(x_1,x_2,\dots,x_n;\theta_1,\theta_2\mid y_1,y_n)=\frac{f(x_1,x_2,\dots,x_n;\theta_1,\theta_2)}{g(y_1,y_n)}$$
$$=\frac{\left(\frac{1}{2\theta_2}\right)^n}{n\,(n-1)\frac{1}{(2\theta_2)^n}\,(y_n-y_1)^{n-2}}$$
$$=\frac{1}{n\,(n-1)\,(y_n-y_1)^{n-2}}=\frac{1}{n\,(n-1)\,(X_{\max}-X_{\min})^{n-2}}$$

$\because$ *which not depend on* θ_1,θ_2

$\therefore$ Y_1 *and* Y_n *are jointly suff est for* θ_1 *and* θ_2 *respectively.*

The Exponential Class of Probability Density Functions

Let X has a p.d.f. $f(x;\theta)$, then the family of $f(x;\theta)$ is belong to exponential class of distribution if it can be expressed as:

$$f(x;\theta)=Exp(\ln f(x;\theta))$$
$$=Exp\left(p(\theta)\,K(x)+S(x)+q(\theta)\right)$$

Such that: p(θ) K(x) must have to be for exponential class.

Ex: Let $X_1, X_2, \dots, X_n$ be a rssn from Bernoulli dist^n Ber(θ), show that if the dist^n of X can be written in exponential form?

Sol:

$$f(x;\theta) = \theta^x (1-\theta)^{1-x}$$
$$= \exp\left(x\ln(\theta) + (1-x)\ln(1-\theta)\right)$$
$$= \exp\left(x\ln\left(\frac{\theta}{1-\theta}\right) + \ln(1-\theta)\right)$$
$$= \exp\left(p(\theta)K(x) + \underset{=0}{S(x)} + q(\theta)\right)$$

$\therefore$ *the family of X is brlongs to the* exp. *class of distribution*

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from Poisson dist^n Poi(θ), show that if the dist^n of X can be written in exponential form?

Sol:

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}$$
$$= \exp\left(\ln f(x;\theta)\right)$$
$$= \exp\left(-\theta + x\ln(\theta) + \ln(x!)\right)$$
$$= \exp\left(q(\theta) + p(\theta)K(x) + S(x)\right)$$

$\therefore$ *the family of X is brlongs to the* exp. *class of distribution*

H.W: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from exponential dist^n Exp(θ), show that if the exponential dist^n belongs to the exponential family?

H.W: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from normal dist^n N(0, θ), show that if the normal dist^n belongs to the exponential family?

Theorem

Let $f(x;\theta)$ belongs to exponential class of distributions, then the j.p.d.f. of $(X_1, X_2, \ldots, X_n)$ is:

$$f(x_1, x_2, \ldots, x_n;\theta) = Exp\left(p(\theta)\sum K(x_i) + \sum S(x_i) + n\,q(\theta)\right)$$

Using factorization theorem then the j.p.d.f. can be written as;

$$f(x_1, x_2, \ldots, x_n;\theta) = Exp\left(p(\theta)\sum K(x_i) + n\,q(\theta)\right)\cdot Exp\left(\sum S(x_i)\right)$$

Then we say that $\sum K(X_i)$ is minimal sufficient estimator for θ.

Ex: In a rss$n$. Find minimal sufficient estimators for parameters of:

1) Poisson(θ). **2)** Beta(α, β).

Sol:

1) $Poisson(\theta)$

$$f(x;\theta)=\frac{e^{-\theta}\,\theta^{x}}{x!}=\exp\left(-\,\theta+x\ln(\theta)-\ln(x!)\right)$$

In a rssn;

$$f(x_1,x_2,\dots,x_n;\theta)\ =\exp\left(-\,n\,\theta+\ln(\theta)\sum x_i\ -\sum\ln(x!)\right)$$

$$=\exp\left(-\,n\,\theta+\ln(\theta)\sum x_i\right)\cdot\exp\left(-\sum\ln(x!)\right)$$

$$\therefore\ f(x_1,x_2,\dots,x_n;\theta)=\exp\left(p(\theta)\sum K(x_i)+n\,q(\theta)\right)\cdot\exp\left(\sum S(x_i)\right)$$

$$\Rightarrow\ \sum K(X_i)=\sum X_i\ \ \textit{is}\ \ \textit{minimal}\ \ \textit{suff}\ \ \textit{est}\ \ \textit{for}\ \ \theta$$

2) $Beta(\alpha,\beta)$

$$f(x;\theta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,x^{\alpha-1}(1-x)^{\beta-1}$$

$$=\exp\left(-\ln\beta(\alpha,\beta)+(\alpha-1)\ln(x)\ +(\beta-1)\ln(1-x)\right)$$

$$=\exp\left(-\ln\beta(\alpha,\beta)+\alpha\ln(x)-\ln(x)+\beta\ln(1-x)-\ln(1-x)\right)$$

In a rssn;

$$f(x_1,x_2,\dots,x_n;\theta)\ =\exp\begin{pmatrix}-\,n\ln\beta(\alpha,\beta)+\alpha\sum\ln(x_i)-\sum\ln(x_i)+\\ +\ \beta\sum\ln(1-x_i)-\sum\ln(1-x_i\ )\end{pmatrix}$$

$$=\exp\left(-\,n\ln\beta(\alpha,\beta)+\alpha\sum\ln(x_i)+\beta\sum\ln(1-x_i)\ \right)\times$$

$$\times\exp\left(-\sum\ln(x_i)-\sum\ln(1-x_i\ )\right)$$

$$=\exp\left(p_1(\alpha)\sum K_1(x_i)+\ p_2(\beta)\sum K_2(x_i)+n\,q(\alpha,\ \beta)\right)\times\exp\left(\sum S(x_i)\right)$$

$$\Rightarrow\ \sum K_1(X_i)=\sum\ln(X_i)\ \ \textit{and}\ \ \sum K_2(X_i)=\sum\ln(1-\ X_i)$$

are min *imal* *suff* *est* *for* α *and* β *respective ly*

Ex: In a rss*n*. Find minimal sufficient estimators for θ, σ^2 from N(θ, σ^2).

Sol:

$$X\ \sim\ N(\theta,\sigma^2)$$

$$f(x;\theta,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{1}{2\sigma^2}(x-\theta)^2}\qquad,\ -\infty<x<\infty$$

$$f(x;\theta,\sigma^2)=\exp\left(\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{1}{2\sigma^2}(x-\theta)^2}\right)\right)$$

$$=\exp\left(-\frac{1}{2}\ln(2\pi\sigma^2)-\frac{(x_i-\theta)^2}{2\sigma^2}\right)$$

$$=\exp\left(-\frac{1}{2}\ln(2\pi\sigma^2)-\frac{x_i^2}{2\sigma^2}+\frac{\theta\,x}{\sigma^2}-\frac{\theta^2}{2\sigma^2}\right)$$

$$f(x;\ \theta,\sigma^2)=\exp\left(p_1(\theta)\,K_1(x)+\ p_2(\sigma^2)\,K_2(x)+q(\theta,\sigma^2)+\ S(x)\right)$$

In a rssn;

$$f(x_1,x_2,\ldots,x_n;\theta,\sigma^2) = \exp\left(-\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum x_i^2}{2\sigma^2} + \frac{\theta\sum x_i}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}\right)$$

$$= \exp\left(p_1(\theta)\sum K_1(x_i) + p_2(\sigma^2)\sum K_2(x_i) + n\,q(\theta,\sigma^2)\right)\times \exp\left(\sum S(x_i)\right)$$

$$\Rightarrow \therefore\ \sum X_i \text{ and } \sum X_i^2 \text{ are minimal jointly suff est for } \theta,\sigma^2 \text{ respectively.}$$

4. Completeness

Let $f(x\,;\theta)$ denote a family of probability density function, **let u(x) be a continuous function of (X), then if [E{u(X)}= 0] implies (u(x) = 0) at each point of (X)**, we say that the family of p.d.f. is complete.

Note: If the range depends on θ, then we use the general rule to derivative of integral;

$$Let;\quad G(\theta) = \int_{a(\theta)}^{b(\theta)} f(x;\theta)\,dx \qquad ,\ where\quad f: is\ any\ function$$

$$\frac{\partial G(\theta)}{\partial\theta} = \int_{a(\theta)}^{b(\theta)} \frac{\partial f(x;\theta)}{\partial\theta}\,dx + f\left(b(\theta),\theta\right)\times b'(\theta) - f\left(a(\theta),\theta\right)\times a'(\theta)$$

Ex: Let X be a random variable from; **1)** Bernoulli dist^n. **2)** Poisson dist^n. **3)** Normal dist^n. Show that the family of X is complete.

Sol: 1)

1) $X \sim Ber(\theta)$

$f(x;\theta) = \theta^x(1-\theta)^{1-x} \quad , x = 0,1$

Let $u(x)$ *be a continuous fun of* X*. then;*

$E\left(u(X)\right) = 0$

$$E\left(u(X)\right) = \sum_{x=0}^{1} u(x)\ f(u;\theta) = 0$$

$$= u(0)\theta^0(1-\theta)^{1-0} + u(1)\theta^1(1-\theta)^{1-1} = 0$$

$$= u(0)(1-\theta) + u(1)\theta = 0$$

$\because \theta \neq 0$

$\therefore u(0) = u(1) = 0 \quad \Rightarrow u(x) = 0\ \ \forall x$

$\therefore$ *the family of* X *is complete*

5) Uniqueness Estimator (M.V.U.E)

Th: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from a dist^n with p.d.f. $f(x\,;\theta)$, let Y_1 be a **sufficient** estimator for θ, and let g(y_1; θ) be **complete** if there is a continuous function of Y_1 which is an **unbiased** estimator for θ, $\phi(\theta)$ such that $E\left(\phi(\theta)\right) = \theta$, **then** $\phi(\theta)$ **is the unique best estimator for θ (M.V.U.E).**

Note: If an estimator does not complete then we do not find the unique and if have complete then we find a unique estimator.

Ex: Let X be a r.v. with p.d.f.;

$$f(x;\theta) = \frac{1}{2\theta} \quad , \quad -\theta < x < \theta \quad , \quad \theta > 0$$

Show that $f(x;\theta)$ is not complete? If it is then find the unique estimator for θ.

Sol:

$\because$ *the range depend on* θ.

Let $u(x)$ *be a continuous fun of* X. *then* ;

$$E(u(X)) = 0 \quad , \quad u(x) = 0$$

$$E(u(X)) = \int_{-\theta}^{\theta} u(x)\, f(x;\theta)\, dx = 0$$

$$\Rightarrow \int_{-\theta}^{\theta} u(x) \frac{1}{2\theta} dx = 0 \qquad \} \times 2\theta$$

$$\Rightarrow \int_{-\theta}^{\theta} u(x)\, dx = 0$$

Let ; $G(\theta) = \int_{a(\theta)}^{b(\theta)} f(x;\theta)\, dx$, *where* f : *is any function*

$$\frac{\partial G(\theta)}{\partial \theta} = \int_{a(\theta)}^{b(\theta)} \frac{\partial f(x;\theta)}{\partial \theta} dx + f(b'(\theta),\theta) \times b'(\theta) - f(a'(\theta),\theta) \times a'(\theta)$$

$$\Rightarrow \frac{\partial}{\partial \theta} \int_{-\theta}^{\theta} u(x)\, dx = 0$$

$$\Rightarrow \int_{-\theta}^{\theta} \frac{\partial\, u(x)}{\partial \theta} dx + u(\theta)(1) - u(-\theta)(-1) = 0$$

$$\Rightarrow \int_{-\theta}^{\theta} (0)\, dx + u(\theta) + u(-\theta) = 0$$

$$\Rightarrow u(\theta) + u(-\theta) = 0$$

If $u(\theta) = -u(-\theta)$ *is odd function*

If $u(\theta) = u(-\theta)$ *is even function*

$\because$ $u(\theta) \neq 0$

$\therefore$ $f(x;\theta)$ *is not complete*

$\because$ $f(x;\theta)$ *is not complete then there isn't has the unique estimator*

Ex: Let $X_1, X_2, \ldots, X_n$ is a rss$n$ from Gamma dist^n $\Gamma(4, \theta)$, $0 < \theta < \infty$. **1)** Show that $Y = \sum X_i$ is a complete sufficient estimator for θ. **2)** Find the unique continuous function of Y which is the best estimator for θ (M.V.U.E).

Sol:

$$\because X \sim \Gamma(4,\theta) \Rightarrow f(x;\theta) = \frac{1}{\Gamma(4)\theta^4} x^{4-1} e^{-x/\theta} \quad , x > 0 \quad , \theta > 0$$

$$= \frac{1}{6\theta^4} x^3 e^{-x/\theta}$$

$\because$ *the range of X does not depend on θ, then we use exponential family to prove suff.*

$$f(x;\theta) = \exp\left(\ln\left\{\frac{1}{6\theta^4} x^3 e^{-x/\theta}\right\}\right) = \exp\left(\ln(1) - \ln(6) - 4\ln(\theta) + 3\ln(x) - \frac{x}{\theta}\right)$$

$$= \exp\left(-\ln(6) - 4\ln(\theta) + 3\ln(x) - \frac{x}{\theta}\right)$$

In arssn

$$f(x_1,...,x_n;\theta) = \exp\left(-n\ln(6) - 4n\ln(\theta) + 3\sum \ln(x_i) - \frac{\sum x_i}{\theta}\right) \qquad , 3\sum \ln(x_i) = \ln\left(\prod_{i=1}^{n} x_i^3\right)$$

$$\therefore f(x_1,...,x_n;\theta) = \exp\left(n\,q(\theta) + p(\theta)\sum K(x_i) + \sum S(x_i)\right)$$

$$\sum K(x_i) = \sum X_i \ , \ p(\theta) = -\frac{1}{\theta} \ , \ \sum S(x_i) = \ln\left(\prod_{i=1}^{n} x_i^3\right) \ , \ n\,q(\theta) = -n\ln(6) - 4n\ln(\theta)$$

$\therefore Y = \sum K(x_i) = \sum X_i$ *is sufficient estimator for θ.*

$$\because X \sim \Gamma(4,\theta) \Rightarrow Y = \sum X_i \sim \Gamma(4n,\theta) \quad , g(y;\theta) = \frac{1}{\Gamma(4n)\,\theta^{4n}} y^{4n-1} e^{-y/\theta} \quad , \quad y > 0$$

Let $u(y)$ be a continuous fun of Y. then ;

$$E(u(Y)) = 0 \qquad , u(y) = 0$$

$$E(u(Y)) = \int_0^\infty u(y)\, g(y;\theta)\, dy = 0$$

$$\Rightarrow \int_0^\infty u(y) \frac{1}{\Gamma(4n)\,\theta^{4n}} y^{4n-1} e^{-y/\theta}\, dy = 0$$

$$\Rightarrow \frac{1}{\Gamma(4n)\,\theta^{4n}} \int_0^\infty u(y)\ y^{4n-1} e^{-y/\theta}\, dy = 0 \quad \} \times \Gamma(4n)\,\theta^{4n}$$

$$\Rightarrow \int_0^\infty u(y)\ y^{4n-1} e^{-y/\theta}\, dy = 0$$

$y^{4n-1} \neq 0$ *(never)* $\quad , e^{-y/\theta} \neq 0 \quad , \Rightarrow \therefore u(y) = 0$

$g(y;\theta)$ *is complete* .

$X \sim \Gamma(\alpha,\beta) \Rightarrow E(X) = \alpha\beta \quad , V(X) = \alpha\beta^2$

$X \sim \Gamma(4,\theta) \Rightarrow E(X) = 4\theta \quad , V(X) = 4\theta^2$

$Y = \sum X_i \sim \Gamma(4n,\theta)$

$E(Y) = 4n\theta \qquad \} \div 4n$

$E\left(\frac{Y}{4n}\right) = \theta \qquad \Rightarrow \therefore \hat{\theta} = \frac{Y}{4n}$ *is M.V.U.E. for θ.*

Ex: (Functions of Parameter): Let $X_1, X_2, \ldots, X_n$ denote a random sample from a dist^n which is Ber(1, θ), find the best estimator for the variance $n\theta(1-\theta)$ of $Y=\sum X_i$ (M.V.U.E).

Sol:

$$X \sim Ber(1,\theta)$$

$$Y = \sum X_i \sim Bin(n,\theta)$$

$$E(Y) = E\left(\sum X_i\right) = nE(X) = n\theta$$

$$\Rightarrow E\left(\frac{Y}{n}\right) = \theta \qquad \Rightarrow \therefore \hat{\theta} = \frac{Y}{n} \text{ is M.V.U.E. for } \theta.$$

But the required is $V(Y) = n\theta(1-\theta)$

$$E\left(V(Y)\right) = E\left(n\frac{Y}{n}\left\{1-\frac{Y}{n}\right\}\right) = E\left(Y-\frac{Y^2}{n}\right) = E(Y) - \frac{E(Y^2)}{n}$$

$$= n\theta - \frac{V(Y)+\left(E(Y)\right)^2}{n} = n\theta - \frac{n\theta(1-\theta)+n^2\theta^2}{n}$$

$$= n\theta - \frac{n\theta - n\theta^2 + n^2\theta^2}{n} = \frac{n^2\theta - n\theta + n\theta^2 - n^2\theta^2}{n}$$

$$= \frac{n\theta(n-1+\theta-n\theta)}{n} = \frac{n\theta(n-1-\theta(n-1))}{n}$$

$$= \frac{n\theta(n-1)(1-\theta)}{n} = n\theta(1-\theta)\frac{(n-1)}{n}$$

$$\therefore E\left(Y\left\{1-\frac{Y}{n}\right\}\right) = n\theta(1-\theta)\frac{(n-1)}{n} \qquad \} \times \frac{n}{n-1}$$

$$E\left(\frac{nY\left(1-\frac{Y}{n}\right)}{n-1}\right) = n\theta(1-\theta)$$

$$\Rightarrow n\hat{\theta}(1-\hat{\theta}) = \frac{nY\left(1-\frac{Y}{n}\right)}{n-1} \text{ is M.V.U.E. for } Y = \sum X_i$$

The Rao- Cramer Inequality

Let $X_1, X_2, \ldots, X_n$ be a rssn from a dist^n with p.d.f. $f(x;\theta)$, and let $T = u(X_1, X_2, \ldots, X_n)$ be an unbiased estimator for $\phi(\theta)$, then the variance of T satisfies the inequality;

$$V(T) \geq \frac{(\phi'(\theta))^2}{nE\left(\frac{\partial \ln f(x;\theta)}{\partial\theta}\right)^2} = \frac{(\phi'(\theta))^2}{Var(S)} = \frac{(\phi'(\theta))^2}{-nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right)}$$

Notes:

1)

$\frac{(\phi'(\theta))^2}{V(S)}$ *is called Rao – Cramer Lower Bound (RCLB) (Minimum variance bound (MVB)*

2) If T unbiased estimator for θ, E(T) = θ;

$\phi(\theta) = \theta \quad \rightarrow \quad \phi'(\theta) = 1$

$$\therefore \left(RCLB = \frac{1}{V(S)} \right)$$

3) In normal distribution case, we apply the second law is easier.
4) We do not use (n) in case using the likelihood function in law.

6. Efficient Estimator

Def^n: The ratio of the RCLB to the actual variance of any unbiased estimator for θ is called the efficiency;

$$eff = \frac{RCLB}{V(T)} \quad , \quad 0 \le eff \le 1$$

$$if \;\; eff = 1 \quad \Rightarrow T \text{ is called efficient estimator for } \theta.$$

Def^n: Let T be an unbiased estimator for $\phi(\theta)$, then we say that T is an efficient estimator for θ iff;

$$V(T) = RCLB$$

Ex: Let $X_1, X_2, \ldots, X_n$ be a rss$n$ from Poison dist^n Poi(θ), if $T = \overline{X}$ is an efficient estimator for $\phi(\theta) = \theta$.

Sol:

$$X \sim Poi(\theta)$$

$$f(x;\theta) = \frac{e^{-\theta}\theta^{x}}{x!} \quad , \quad x = 0, 1, \ldots\ldots$$

$$\ln f(x;\theta) = -\theta + x\ln(\theta) - \ln(x!)$$

$$\frac{\partial \ln f(x;\theta)}{\partial \theta} = -1 + \frac{x}{\theta} = \frac{x-\theta}{\theta}$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = -\frac{x}{\theta^2}$$

$$-E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right) = \frac{E(X)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$$-nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right) = \frac{n}{\theta} = V(S)$$

$$\phi(\theta) = \theta \quad \rightarrow \quad \phi'(\theta) = 1$$

$$RCLB = \frac{(\phi'(\theta))^2}{V(S)} = \frac{1}{\frac{n}{\theta}} = \frac{\theta}{n}$$

$$V(T) = V(\overline{X})$$

$$V(\overline{X}) = \frac{V(X)}{n} = \frac{\theta}{n}$$

$$\therefore RCLB = V(\overline{X}) \quad \Rightarrow \quad eff = 1$$

$$\therefore \overline{X} \text{ is an efficient estimator for } \phi(\theta).$$

Ex: In a rss$n$ from N(θ, σ^2). Show that;

1) If $T=\overline{X}$ is an efficient estimator for $\phi(\theta)=\theta$.

2) $S^2=\dfrac{\sum(x_i-\overline{x})^2}{n}$ or $S^2=\dfrac{\sum(x_i-\overline{x})^2}{n-1}$ is an efficient estimator for $\phi(\sigma^2)=\sigma^2$.

Sol:

1) $X\sim N(\theta,\sigma^2)$

$$f(x;\theta,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{1}{2\sigma^2}(x-\theta)^2}\quad,\ -\infty<x<\infty$$

$$\ln f(x;\theta,\sigma^2)=-\frac{1}{2}\ln(2\pi\sigma^2)-\frac{1}{2\sigma^2}(x-\theta)^2$$

$$\frac{\partial\ln f(x;\theta,\sigma^2)}{\partial\theta}=zero+\frac{2}{2\sigma^2}(x-\theta)=\frac{x-\theta}{\sigma^2}$$

$$\frac{\partial^2\ln f(x;\theta,\sigma^2)}{\partial\theta^2}=\frac{-1}{\sigma^2}$$

$$-n\,E\left(\frac{\partial^2\ln f(x;\theta,\sigma^2)}{\partial\theta^2}\right)=\frac{n}{\sigma^2}=V(S)=F.I$$

$$\phi(\theta)=\theta\ \rightarrow\ \phi'(\theta)=1$$

$$RCLB=\frac{(\phi'(\theta))^2}{V(S)}=\frac{1}{n/\sigma^2}=\frac{\sigma^2}{n}$$

$$V(T)=V(\overline{X})$$

$$V(\overline{X})=\frac{V(X)}{n}=\frac{\sigma^2}{n}$$

$$\therefore RCLB=V(\overline{X})=\frac{\sigma^2}{n}\quad,\ \Rightarrow eff=1$$

$\therefore \overline{X}$ *is an efficient estimator for* $\phi(\theta)=\theta$.

Mean Square Error (MSE)

One way of measuring the accuracy of an estimator is via its mean square error. The mean square error of an estimator $\hat{\theta}$ is defined as:

$$MSE(\hat{\theta})=E(\hat{\theta}-\theta)^2=Var(\hat{\theta})+b^2(\hat{\theta})$$

Note: If $\hat{\theta}$ is unbiased estimator for θ then; $MSE(\hat{\theta})=Var(\hat{\theta})$

Relative Efficient Estimator

Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimators for parameter θ of $f(x\,;\,\theta)$, the relative efficient of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is given by:

$$R.Eff\,.(\hat{\theta}_1 \mid \hat{\theta}_2) = \frac{MSE\ (\hat{\theta}_1)}{MSE\ (\hat{\theta}_2)} < 1$$

$i.e.,\quad MSE\ (\hat{\theta}_1) < MSE\ (\hat{\theta}_2)$

$\therefore\ \hat{\theta}_1$ *is more efficient than* $\hat{\theta}_2$.

Ex: In a rss2 from Bernoulli dist^n Ber(θ), let $T_1 = X_1$ and $T_2 = \frac{\sum X_i}{n+1}$ be two estimators for parameter θ, show that which of them more efficient.

Sol:

$E(T_1) = E(X_1) = \theta \quad unbiased$

$$E(T_2) = E\left(\frac{\sum X_i}{n+1}\right) = \frac{n}{n+1}\,E(X) \quad , \ when \quad n = 2$$

$$E(T_2) = \frac{2}{3}\,\theta \quad biased$$

$$b(T_2) = E(T_2) - \theta = \frac{2}{3}\,\theta - \theta = \frac{-\,\theta}{3}$$

$V(T_1) = Var(X_1) = \theta(1-\theta) = MSE(T_1)$

$$V(T_2) = \frac{1}{(n+1)^2} Var(\sum X_i) = \frac{1}{(n+1)^2}\ n\,Var(X) = \frac{2}{9}\theta(1-\theta) = \frac{2\theta - 2\theta^2}{9}$$

$MSE(T_2) = V(T_2) + b^2(T_2)$

$$= \frac{2\theta - 2\theta^2}{9} + \frac{\theta^2}{9} = \frac{2\theta - \theta^2}{9}$$

$$\because\ \frac{2\theta - \theta^2}{9} < \theta(1-\theta)$$

$\therefore MSE(T_2) < MSE(T_1)$

$\Rightarrow\ \therefore\ T_2$ *is more efficient than* T_1.