Matrices

Administration & Economic of College (2023- 2024)

Second Stage

Department of Statistics

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First Semester

<u>Course Reading List and References:</u>

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٥. ايزو، الدكتور فرانك ،ترجمة(نخبة من الاساتذة المتخصصين)، ملخصات شوم نظريات ومسائل في المصفوفات، ١٩٦٢.

٦. الناصر، عبد المجيد حمزة ، جواد، لميعة باقر، الجبر الخطي، تموز ١٩٨٨.

Chapter One “1“

Matrices :(المصفوفات) (ريزكراوه)

Matrix: An (m×n) real (complex) matrix A is an array of real (complex) numbers a_{ij} $(1 \le i \le m\ ,\ 1 \le j \le n)$ arranged in (m) rows and (n) columns, and enclosed by brackets, as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Notes:

a) If A is (m×n) , then m is number of rows in the array.
b) If A is (m×n) , then n is number of columns in the array.
c) The size (or order) of the matrix is (m×n).
d) The a_{ij} appears in the ith rows and jth columns.
e) The numbers of are called the elements of the matrix.
f) The notation is sometimes abbreviated to $[a_{ij}]$, or $[a_{ij}]_{m\times n}$ if we wish to specify the size of the array.

Example 1: Here are some various matrices with different sizes:

$$\boldsymbol{A} = \begin{bmatrix} -5 & 10 & 0 & 1 \\ 3 & 2 & -5 & 1 \end{bmatrix} \quad \text{(Size: } 2\times 4\text{)}$$

$$B = \begin{bmatrix} 100 & 2 \\ 1/5 & -1 \\ 1 & -2/5 \end{bmatrix} \quad \text{(Size: } 3\times 2\text{)}$$

$C = \begin{bmatrix} 3 & -4 & 8 & 1 \end{bmatrix}$ (Size: 1×4)

$D = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ (Size: 3×1)

Example2:

$A=\begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 6 \end{bmatrix}_{2\times 3}$ is a 2×3 matrix in which $a_{11}=1$, $a_{12}=2$, $a_{13}=-3$, $a_{21}=4$, $a_{22}=0$, $a_{23}=6$

$A=\begin{bmatrix} 1 & -2 \\ 3 & 4 \\ 5 & 7 \end{bmatrix}_{3\times 2}$ is a 3×2 matrix in which $a_{11}=1$, $a_{12}=-2$, $a_{21}=3$, $a_{22}=4$, $a_{31}=5$, $a_{32}=7$

Note: **قد تشكل عناصر المصفوفة في بعض الاحيان دالة وفي هذه الحالة يمكن استخراج جميع العناصر بسهولة**

Example:

Find the elements of matrix A=(a_{ij}) for size 3×2. Where $a_{ij}=i^2+3j$

Solution:

$$A_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$$

$a_{ij}=i^2+3j$

$a_{11}= (1)^2+ 3(1) = 1+3 = 4$

$a_{12}= (1)^2+ 3(2) = 1+6 = 7$

$a_{21}= (2)^2+ 3(1) = 4+3 = 7$

$a_{22}= (2)^2+ 3(2) = 4+6 = 10$

$a_{31}= (3)^2+ 3(1) = 9+3 = 12$

$a_{32}= (3)^2+ 3(2) = 9+6 = 15$

$$\therefore A=\begin{bmatrix} 4 & 7 \\ 7 & 10 \\ 12 & 15 \end{bmatrix}$$

Square matrix:(المصفوفة المربعة) ریزكراوەی دووجا

If number of rows (m) and columns (n) in any matrix are equal (m=n) we said this matrix is Square matrix.

Or An (m×n) matrix A is Square if (m=n), that is if A has the same number of rows and columns. In a Square matrix $A=[a_{ij}]_{m\times n}$, a_{11}, a_{22} , … , a_{nn} are called the element of the main (سەرەكی) (or leading بنچینە) diagonal(لاداو، لیژ). Or A_n

Examples:

$A=\begin{bmatrix} 2 & -1 \\ 3 & 8 \end{bmatrix}_{2\times 2}$ (2× 2 square matrix of order 2)

$B=\begin{bmatrix} 2 & 5 & 4 \\ -3 & 2 & 7 \\ 0 & -6 & 9 \end{bmatrix}_{3\times 3}$ (3× 3 square matrix of order 3)

$C=\begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}_{2\times 2}$ (2× 2 square matrix of order 2)

$D=\begin{bmatrix} -8 & 1 & 0 \\ 2 & 3 & 1 \\ 1/2 & 0 & 10 \end{bmatrix}_{3\times 3}$ (3×3 square matrix of order 3).

Equal matrix: (المصفوفة المتساوية) ریزكراوی یەكسان

Two matrices $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{r\times s}$ are equal if (m=r , n=s) and $a_{ij}= b_{ij}$ $1\le i\le m(=r)$, $1\le j\le n(=s)$; that is , if they have the same number of rows, the same number of columns, and corresponding elements are equal. *or* Two matrix equal if and only if they have exactly the same elements.

Example1:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \Leftrightarrow a=2,\ b=0,\ c=1 \text{ and } d=3$$

The properties of equal matrix:

1) A=A for all matrix A.
2) A=B then B=A for all A , B matrix.
3) If A=B and B=C then A=C for all A, B C matrix.

Zero matrix: (المصفوفة الصفرية) ريزكراوى سفر

A is matrix if all elements are zero then A is called zero matrix (A=0). We denote such a matrix by ($0_{m\times n}$) or simply by($\underline{0}$) . If there can be confusion about its size.

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Then:

- A+ 0 = A,
- A – A = 0,
- 0 A = 0, A 0 = 0

Example1:

The following examples represent 2×2, 3×2, 2×4 zero matrices

$$\underset{2\times2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}},\quad \underset{3\times2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}},\quad \underset{2\times4}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Algebraic operations: (العمليات الجبرية) كردارەكاني بيركارى

1- Addition of matrices:(جمع المصفوفات) كردارى كۆكردنەوەى ريزكراوەكان

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two (m×n) matrices, their sum (A+B) is defined to be the matrix $[\,a_{ij} + b_{ij}\,]_{m\times n}$, where C=A+B

Mathematically, we express $C = A + B$ as

$$[c_{ij}]_{m\times n} = [\,a_{ij} + b_{ij}\,]_{m\times n}$$

For Example1:let A and B are two matrices,

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \quad , \quad B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times3}$$

summing the two matrices yields

$$C = A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}_{3\times3}$$

تێبینی: واتە کرداری کۆکردنەوەی دوو ریزکراو لـەکاتێك ئەنجام دەدرێت ئەگەر هاتوو هەردوو ریزکراو هەمان قەبارەیان هەبێت.

Example1:

Given the 2×3 matrices $\boldsymbol{A}=\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$, we see that

$$\boldsymbol{A}+\boldsymbol{B}=\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}+\begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}=\begin{bmatrix} 2+2 & 1+(-3) & 1+4 \\ -1+(-3) & -1+1 & 4+(-2) \end{bmatrix}=\begin{bmatrix} 4 & -2 & 5 \\ -4 & 0 & 2 \end{bmatrix}$$

Example2:

Given the 2×2 matrix $\boldsymbol{A}=\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ and the 3×2 matrix $\boldsymbol{B}=\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$, the matrix $\boldsymbol{A}+\boldsymbol{B}$ is not defined since A and $\boldsymbol{B}$ are not of the same size.

Example3:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix} \quad \Rightarrow \quad A+B = \begin{bmatrix} 6 & 6 \\ 7 & 5 \\ 1 & 1 \end{bmatrix}$$

Example4:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix} \quad \Rightarrow \quad A+C$$ is undefined (پێناسە نەکراوە–نەزانراوە).

Example5:

Given the 2×2 matrices $\begin{bmatrix} 9 & -3 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ -1 & 6 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix}$,

find: A + B + C = D i.e. $a_{ij} + b_{ij} + c_{ij} = d_{ij}$

$$\begin{bmatrix} 9 & -3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 2 \\ 6 & 5 \end{bmatrix}$$

The properties: (الخصائص الجمع) تایبەتمەندییەکانی کۆکردنەوە

The properties of Addition of matrices:

Let it all of A, B and C are the matrices acceptable (گونجاو) (suitable) for addition in size (m×n) then the law is correct:

1. Commutative law (قانون التبديل) یاسای جێگۆرکێ

   A + B = B + A

2. Associative law: (قانون التجميع) یاسای کۆکراوە

   (A + B) + C = A + (B + C)

3. A + <u>0</u> = <u>0</u> + A = A
4. A + (-A) = -A + A = <u>0</u>
5. *if* A=B A + C = B + C

   If A + C = B + C then A=B

Proof:

1. A + B = B + A , A=(a_{ij}) , B=(b_{ij})

   A + B = (a_{ij}) + (b_{ij})

   = (a_{ij} + b_{ij})

   = (b_{ij} + a_{ij})

   = (b_{ij}) + (a_{ij})

   = B + A

2. (A + B) + C = A + (B + C)

Subtraction of matrices: (طرح المصفوفات) کرداری لێدەرکردنی ریزکراوەکان

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two (m×n) matrices, their Subtraction (A-B) is defined to be the matrix $[a_{ij} - b_{ij}]_{m\times n}$, where $A - B = A + (-B)$

Mathematically, we express $C = A - B$ as

$$[c_{ij}]_{m\times n} = [a_{ij} - b_{ij}]_{m\times n}$$

For Example1:let A and B are two matrices,

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}, \quad B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times 3}$$

Subtracting the two matrices yields

$$C = A - B = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{bmatrix}_{3\times 3}$$

Example1: Let $A=\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}_{3\times 2}, \quad B=\begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix}_{3\times 2} \Rightarrow$ *find* $A-B$ *and* $B-A$

Solution:

$$A-B=\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0-6 & 1-5 \\ 3-4 & 2-3 \\ -1-2 & 0-1 \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ -1 & -1 \\ -3 & -1 \end{bmatrix}$$

$$B-A=\begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6-0 & 5-1 \\ 4-3 & 3-2 \\ 2-(-1) & 1-0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$$

H.W: Let $A=\begin{bmatrix} 1 & 4 & -2 \\ 3 & 4 & 5 \end{bmatrix}, \quad B=\begin{bmatrix} -2 & 4 & 3 \\ 2 & 5 & -1 \end{bmatrix}$. Find all the:

1) A + B

2) A – B

2- <u>Multiplication of a matrix by a scalar:</u> (

Let $A=(a_{ij})$ of a matrix of size (m×n) and (k) be a scalar number then:

$k.A = A.k = (k\, a_{ij})$

<u>Example1:</u>

Let $A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 7 & 0 \end{bmatrix}_{3\times2}$ and k=3, then find k.A?

$$k.A = 3\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 3(2) & 3(3) \\ 3(-1) & 3(4) \\ 3(7) & 3(0) \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -3 & 12 \\ 21 & 0 \end{bmatrix}_{3\times2}$$

<u>The law for Multiplication of a matrix by a scalar:</u>

Let A, B be the matrix for size (m×n) and α (الفا), β (بيتا) are scalar, then:

1) $1.A = A.1 = A$
2) $\alpha (A+B) = \alpha A + \alpha B$
3) $(\alpha + \beta) A = \alpha A + \beta A$
4) $\alpha (\beta A) = (\alpha \beta)A$
5) $\underline{0} .A = A. \underline{0} = \underline{0}$

<u>H.W</u>: Let $A=\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Find all the:

1) 2 (A+B).
2) 3 (2A).
3) $\underline{0}$ .B
4) 2 A + 4B
5) B – 2A.
6) -3A + B.
7) $\underline{0}$ .A – 2B.
8) (-4+ 7)B + 5A.
9) 3(B – A).
10) 4B – 3A

4- <u>Multiplication of two matrices:</u> (ضرب المصفوفات) کرداری جاران کردنی ریزکراوەکان

Let $A=(a_{ij})$ for size (m×n) and $B=(b_{ij})$ for size (n×p), then A.B is defined if and only if (n=n).

$A_{m\times n} . B_{n\times p} = C_{m\times p}$

Then:

$$C = \underset{m\times n}{A} \cdot \underset{n\times p}{B}$$

$m\times n$ <——> $n\times p$ (*must equal*)

Size of Product $\boldsymbol{m\times p}$

- ***Notes***: This product is defined only when the number of columns of matrix A is equal to the number of rows of matrix B .

Mathematically, if C is a matrix resulting from the multiplication of two matrices, *A* and *B*, then the elements (c_{ij}) of *C* are given by:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad \rightarrow \text{ Equation }_1$$

where k = 1, 2,…, n is the number of columns in *A and* the number of rows in *B*. Look carefully at the subscripts of *a* and *b*, and note that Equation 1 *requires* that the number of columns in the left-hand matrix *must be the same as* the number of rows in the right-hand matrix. Also note that Equation 1 tells us that the product matrix has *i* rows and *j* columns.

What does Equation 1 mean? Well, if we wish to calculate the product of two matrices *A* and *B*:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

then $n = 3$, and the product $C = AB$ is defined by Equation 1 as:

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

For example, suppose you define the (matrix *C*) as the product of the two 3×3 matrices, *A* and *B*, shown above. If you wish to calculate the value of c_{11},

$$\begin{bmatrix} \textcircled{c_{11}} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

you work element-by-element across the first row of the left-hand matrix and element-by-element down the first column of the right-hand matrix as follows:

$$c_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \Rightarrow c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

Similarly, to calculate the value of c_{23}

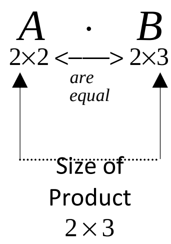
$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

you work across the second row of the left-hand matrix and down the third column of the right-hand matrix:

$$c_{23} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \Rightarrow c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$$

Example2:

If $\boldsymbol{A} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix}$, then the size (2×3) of the product AB is obtained by observing:



The 2×3 matrix AB is computing by performing the following row column multiplications:

$$AB = \begin{bmatrix} (row\ 1\ A)\cdot(column\ 1\ B) & (row\ 1\ A)\cdot(column\ 2\ B) & (row\ 1\ A)\cdot(column\ 3\ B) \\ (row\ 2\ A)\cdot(column\ 1\ B) & (row\ 2\ A)\cdot(column\ 2\ B) & (row\ 2\ A)\cdot(column\ 3\ B) \end{bmatrix}.$$

Performing these multiplications, we obtain:

$$\boldsymbol{AB} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} (1)(1)+(-3)(-1) & (1)(0)+(-3)(2) & (1)(-2)+(-3)(3) \\ (0)(1)+(2)(-1) & (0)(0)+(2)(2) & (0)(-2)+(2)(3) \end{bmatrix} = \begin{bmatrix} 4 & -6 & -11 \\ -2 & 4 & 6 \end{bmatrix}.$$

Example3:

find the product of two matrices, A and B. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution: We first note that multiplication of A by B is allowed by Equation $_1$ because the number of columns in A is the same as the number of rows in B, which allows us to calculate $C = AB$ as:

$$C = AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1*4+2*7+3*1 & 1*5+2*8+3*2 & 1*6+2*9+3*3 \\ 4*4+5*7+6*1 & 4*5+5*8+6*2 & 4*6+5*9+6*3 \\ 7*4+8*7+9*1 & 7*5+8*8+9*2 & 7*6+8*9+9*3 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 27 & 33 \\ 57 & 72 & 87 \\ 93 & 117 & 141 \end{bmatrix}$$

Laws of multiplication of matrices:

Let A, B and C be any real (complex) matrices, and let (α) be any real number. When all the following sums and products are defined, matrix multiplication satisfies the following properties:

1- (A.B)C = A(B.C) (matrix multiplication is associative)
2- A(B+C) = A.B + A.C
3- (A+B)C = A.C + B.C
4- α (A.B) = (αA)B = A(αB)
5- $A.B \neq B.A$ (بشكل عام) بە شێوەیەکی گشتی
6- If A.B = 0 does not necessarily imply that $A \neq 0$, $B \neq 0$.
7- If A.B = A.C does not necessarily imply that B = C.

Example1:

If $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}_{2\times 2}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}_{2\times 2}$ and $C = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}_{2\times 2}$, then find:

1) $(A\times B)\times C = A\times(B\times C)$

Then: $(A\times B)\times C = A\times(B\times C)$

2) $A\times(B+C) = (A\times B)+(A\times C)$

Power of the matrices: (رفع المصفوفات) لە ریزکراوەکان (توان) کرداری بەرزکردنەوە

Let A is a Square matrix and (r) is positive real number then:

1- A^r= A.A. … A

For Example: $A^2 = A \cdot A$

$A^3 = A \cdot A \cdot A$

2- (A.B) ^{r} = A^r. B^r if A.B=B.A

3- In general $(A \pm B)^2 \neq A^2 \pm 2A.B + B^2$

4- If and only if A.B=B.A then $(A \pm B)^2 = A^2 \pm 2A.B + B^2$

5- If and only if A.B=B.A then $A^2 - B^2 = (A - B)\ (A + B)$

Example1: let $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 3 \\ 2 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 2 & 2 \end{bmatrix}$ find all:

1- A.B
2- A^2
3- $(A.B)^2$

Example3: If A.B = B.A , show that $(A.B)^4 = A^4.B^4$

Solution:

$(A.B)^4$ = (A.B) (A.B) (A.B) (A.B)
= A.B A.B A.B A.B , ∵ A.B = B.A
= A.A.B.B.A.B.A.B , ∵ A.B = B.A
= A.A.B.A.B.B.A.B , 〃
= A.AA.B.B.B.A.B , 〃
= A.AA.B.B.A.B.B , 〃
= A.AA.B.A.B.B.B , 〃
= A.AA.A.B.B.B.B , 〃

Then: $(A.B)^4 = A^4.B^4$

Example4: If $A = [3 \quad -1 \quad 4]_{1\times3}$ and $B = \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}_{3\times1}$, find A.B , A^2

Solution: The number of columns in A is the same as the number of rows in *B*, which allows us to calculate *(A . B)* as:

$$AB = [3 \quad -1 \quad 4]\begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix} = (3)\times(-2) + (-1)\times(6) + 4\times(3) = [-6 + (-6) + 12] = 0$$

A^2 = A .A

$$\underset{1\times3}{A} \cdot \underset{1\times3}{A} \quad (1\times3 \underset{\text{not equal}}{<\!\!-\!\!-\!\!>} 1\times3)$$

that the product A^2 does not exist (the A is not a Square matrix).

Example5: For the matrices, $\boldsymbol{A}=\begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 0\end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix}1 & -1 & 0\\ 2 & -5 & 1\end{bmatrix}$, after observing from

$\underset{3\times 2}{\boldsymbol{A}} \cdot \underset{2\times 3}{\boldsymbol{B}}$ (3×2 <---> 2×3: *are equal*)

Size of Product 3×3

$$AB=\begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 2 & -5 & 1\end{bmatrix}=\begin{bmatrix}(3)(1)+(0)(2) & (3)(-1)+(0)(-5) & (3)(0)+(0)(1)\\ (1)(1)+(1)(2) & (1)(-1)+(1)(-5) & (1)(0)+(1)(1)\\ (-1)(1)+(0)(2) & (-1)(-1)+(0)(-5) & (-1)(0)+(0)(1)\end{bmatrix}=\begin{bmatrix}3 & -3 & 0\\ 3 & -6 & 1\\ -1 & 1 & 0\end{bmatrix}_{3\times 3}.$$

To compute the product $\boldsymbol{BA}$, we observe

$\underset{2\times 3}{\boldsymbol{B}} \cdot \underset{3\times 2}{\boldsymbol{A}}$ (2×3 <---> 3×2: *are equal*)

Size of Product 2×2

$$BA=\begin{bmatrix}1 & -1 & 0\\ 2 & -5 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}(1)(3)+(-1)(1)+(0)(-1) & (1)(0)+(-1)(1)+(0)(0)\\ (2)(3)+(-5)(1)+(1)(-1) & (2)(0)+(-5)(1)+(1)(0)\end{bmatrix}=\begin{bmatrix}2 & -1\\ 0 & -5\end{bmatrix}_{2\times 2}.$$

Where: $AB\neq BA$

Example5: The matrices $\boldsymbol{A}=\begin{bmatrix}1 & -1 & 0 & 2\\ 2 & -5 & 1 & 5\end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 0\end{bmatrix}$, one can immediately see by observing

$\underset{2\times 4}{\boldsymbol{A}} \cdot \underset{3\times 2}{\boldsymbol{B}}$ (2×4 <—> 3×2: *not equal*)

that the product $\boldsymbol{AB}$ does not exist (the number of columns in the left matrix $\boldsymbol{A}$ (4) is not equal to the number or rows in the right matrix $\boldsymbol{B}$ (3). However, by seeing

$\underset{3\times 2}{B} \cdot \underset{2\times 4}{A}$ (3×2 <—> 2×4: *are equal*)

Size of Product 3×4

the product $\boldsymbol{BA}$ is the 3×4 matrix given by

$$BA=\begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 0 & 2\\ 2 & -5 & 1 & 5\end{bmatrix}=\begin{bmatrix}3 & -3 & 0 & 6\\ 3 & -6 & 1 & 7\\ -1 & 1 & 0 & -2\end{bmatrix}_{3\times 4}.$$

H.W$_1$: If $A=\begin{bmatrix}2 & -4\\ 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}5 & 3\\ 2 & 1\end{bmatrix}$ and $C=\begin{bmatrix}4 & 3\\ 2 & 4\end{bmatrix}$, then find:

1. A(B+C) = A.B + A.C
2. α (A.B) = (αA)B = A(αB) , let α = 3
3. $A.B \neq B.A$
4. (A.B)C = A(B.C)
5. (A+B)C = A.C + B.C
6. $A.^2$, B^2, $C.^2$
7. $(A.B)^2$, $(A.C)^2$, $(B.C)^2$

H.W₂: let $A=\begin{bmatrix}-4 & 0\\ 2 & 0\end{bmatrix}$, $B=\begin{bmatrix}6 & -1\\ 5 & 2\end{bmatrix}$, $C=\begin{bmatrix}6 & -1\\ 4 & 9\end{bmatrix}$, Show that A.B=A.C

Type of matrices: (أنواع المصفوفات) جۆرەکانی ریزکراوە

1. **Diagonal matrix:** (المصفوفة القطرية) (لیێژ) ریزکراوی لارەهیّل

A square matrix $A=[a_{ij}]_{n\times n}$ is called *diagonal matrix* if all non-diagonal entries are zero [$a_{ij}=0$ for all $i \neq j$]. We write diagonal matrix diag(a_{11} , a_{22} , … , a_{nn})

$$A=\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0\\ 0 & a_{22} & 0 & \cdots & 0\\ 0 & 0 & a_{33} & 0\cdots & 0\\ & & & & \vdots\\ 0 & 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \cdots 0 & a_{nn}\end{bmatrix}$$

Diagonal Matrix

Example: $A=\begin{bmatrix}1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 2\end{bmatrix}_{3\times 3}$, $B=\begin{bmatrix}5 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}_{4\times 4}$

- For any $(m\times n)$ matrix A , $I_m A = AI_n = A$.

Note that ***I*** is always a square matrix, that is, the number of rows equals the number of columns. Of course, the size of ***I*** is dependent on the size of ***A*** when multiplying on the left and right as the next example demonstrates.

Example2: Let $B=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, Show that ***B*** is idempotent matrix.

Solution: $B^2 = B\times B = \left[\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\times\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right] = \left[\begin{pmatrix}(1\times1)+(0\times1) & (1\times0)+(0\times0)\\ (1\times1)+(0\times1) & (1\times0)+(0\times0)\end{pmatrix}\right] = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}_{2\times2} = B$

H.W: $A=\begin{bmatrix}5 & -8 & -4\\ 3 & -5 & -3\\ -1 & 2 & 2\end{bmatrix}$ show that $A^2 = A$.

2. **Nilpotent matrix:** (مصفوفة معدومة القوى) ریزکراوی بیٚ هێز

We called for $A=[a_{ij}]_{n\times n}$ Nilpotent matrix when: ($A^2 = 0$).

Example1: let $A=\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}$, Show that $A^2 = \underline{0}$

Solution:

$$A\times A = \left[\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\times\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\right] = \left[\begin{pmatrix}(-1\times-1)+(-1\times1) & (-1\times-1)+(-1\times1)\\ (1\times-1)+(1\times1) & (1\times-1)+(1\times1)\end{pmatrix}\right] = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}_{2\times2} = 0$$

Example2: Let $A=\begin{bmatrix}1 & -3 & -4\\ -1 & 3 & 4\\ 1 & -3 & -4\end{bmatrix}$ show that ***A*** is nilpotent matrix.

H.W: If $A = \begin{bmatrix}2 & -4\\ 1 & -2\end{bmatrix}$, is ***A*** nilpotent matrix?

The properties of square matrix, diagonal matrix and Identity matrix:

1. Let A is a matrix for any number (size):

   A.I = I.A = A that is:

   a) If $A= [a_{ij}]_{n\times n}$ then $A . I_n= I_n . A = A$

   b) If $A= [a_{ij}]_{m\times n}$ then $I_m . A = A . I_n = A$ (See page$_{24\text{-}25}$)

2. $I^r = I . I . I \ldots I = I$
3. If $A= [a_{ij}]_{n\times n}$ and S is a scalar matrix for the same size then A . S = S . A
4. If $A= [a_{ij}]_{n\times n}$ and D is a diagonal matrix not scalar matrix for the same size then:

$$A . D \neq D . A$$

5. If A and B are the diagonal matrix then: A . B = B . A = diagonal matrix.

Example1:a) Let $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$, show that $A\,I_n = I_n\,A = A$.

Solution:

$$A.\,I_2 = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} = A$$

and

$$I_2 .\, A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} = A$$

$$\therefore\; A\,I_n = I_n\,A = A$$

Trace of matrix: شوێنەوارى (شوێن پێ) ریزكراو (أثر المصفوفة)

Is a sum for element main diagonal in square matrix. When:

If $A = [a_{ij}]_{n\times n}$ then:

$$tr(A) = a_{11} + a_{22} + a_{33} + \ldots + a_{nn}$$

$$= \sum_{i=1}^{n} a_{ii}$$

Chapter two

Partitioning of matrix and partition algebraic processes:تجزئة المصفوفات وعمليات الجبرية بطريقة التجزئة

- **Partitioning of matrixتجزئة المصفوفات:**
  It could be partined(تفريق، فصل) any matrix to the small partitions named with (sub matrices), by doing vertical and horizontal lines among the matrix columns and rows it capital symbolized (يرمز الى) to the sub matrix with capital letters first one for rows and the second for columns.

يمكن تجزئة أي مصفوفة بأمرار خطوط افقية و عمودية بين صفوف و أعمدة المصفوفة فتقسم الى أجزاء تسمى مصفوفات جزئيةsub matrices) ويرمز لها بحروف كبيرة مؤشرة بمؤشريين الاول لصفوف و الثاني للأعمدة فتكون) (A_{ij} , B_{ij} , C_{ij} , ...

ex: $A=\begin{bmatrix} 5 & 3 & -2 & 2 \\ 0 & 7 & 3 & 2 \\ -2 & -1 & 0 & 6 \end{bmatrix}$

- **Addition of matrices by partition:(الجمع المصفوفة بالتجزئة)**
  ملاحظة: يتم جمع المصفوفتين إذا كان من نفس الدرجة ومجزئة بنفس الشكل.

  دوو ڕیزکراوە کۆدەکرێنەوە ئەگەر بێت و هەمان قەبارەیان هەبێت وە وەك یەکتر بەش کرابن.

  Let A= ((a$_{ij}$)) and B= ((b$_{ij}$)) of the order (m×n). and let A is partition by:

$$A=\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{sr} \end{bmatrix}$$

and the order of each sub matrices is: $\begin{bmatrix} m_1\times n_1 & m_1\times n_2 & \dots & m_1\times n_r \\ m_2\times n_1 & m_2\times n_2 & \cdots & m_2\times n_r \\ \vdots & \vdots & \ddots & \vdots \\ m_s\times n_1 & m_s\times n_2 & \cdots & m_s\times n_r \end{bmatrix}$

and let B is partition by:

$$B=\begin{bmatrix} B_{11} & B_{12} & \dots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{sr} \end{bmatrix}$$

and the order of each sub matrices is: $\begin{bmatrix} m_1\times n_1 & m_1\times n_2 & \dots & m_1\times n_r \\ m_2\times n_1 & m_2\times n_2 & \cdots & m_2\times n_r \\ \vdots & \vdots & \ddots & \vdots \\ m_s\times n_1 & m_s\times n_2 & \cdots & m_s\times n_r \end{bmatrix}$

then A + B= $\begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \dots & A_{1r}+B_{1r} \\ A_{21}+B_{21} & A_{22}+B_{22} & \cdots & A_{2r}+B_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1}+B_{s1} & A_{s2}+B_{s2} & \cdots & A_{sr}+B_{sr} \end{bmatrix}$ **of size (m×n)**

Multiplication matrices by partition:(ضرب المصفوفة بالتجزئة)

ملاحظة: يتم ضرب المصفوفتين إذا كان عدد الاعمدة في المصفوفة الاولى مساوِي، الى عدد الصفوف في المصفوفة الثانية.

Let A=(a_{ij}) for size (m×p) and B=(b_{ij}) for size (p×n), then A.B is defined if and only if, and

matrix A is partined by:

$$A=\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{sr} \end{bmatrix} = \begin{bmatrix} m_1\times p_1 & m_1\times p_2 & \dots & m_1\times p_r \\ m_2\times p_1 & m_2\times p_2 & \cdots & m_2\times p_r \\ \vdots & \vdots & \ddots & \vdots \\ m_s\times p_1 & m_s\times p_2 & \cdots & m_s\times p_r \end{bmatrix}$$

and matrix B is partined by:

$$B=\begin{bmatrix} B_{11} & B_{12} & \dots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rk} \end{bmatrix} = \begin{bmatrix} p_1\times n_1 & p_1\times n_2 & \dots & p_1\times n_k \\ p_2\times n_1 & p_2\times n_2 & \cdots & p_2\times n_k \\ \vdots & \vdots & \ddots & \vdots \\ p_s\times n_1 & p_s\times n_2 & \cdots & p_s\times n_k \end{bmatrix}$$

ملاحظة: لكي يتم ضرب المصفوفة يجب أن تكون الخطوط العمودية للمصفوفة الاولى مساوية للخطوط الافقية للمصفوفه الثانية. فإن $A\times B$ يكون بشكل التالي:

$$A\,B=\begin{bmatrix} A_{11}B_{11}+A_{12}B_{21}+\ldots+A_{1r}B_{r1}, & \ldots, A_{11}B_{1k}+A_{12}B_{2k}+ & \dots & +A_{1r}B_{rk} \\ A_{21}B_{11}+A_{22}B_{21}+\ldots+A_{2r}B_{r1}, & \ldots, A_{21}B_{1k}+A_{22}B_{2k}+ & \cdots & +A_{2r}B_{rk} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1}B_{11}+A_{s2}B_{21}+\ldots+A_{sr}B_{r1}, & \ldots, A_{s1}B_{1k}+A_{s2}B_{2k}+ & \cdots & A_{sr}B_{rk} \end{bmatrix}$$

Or:

$$A\times B=\begin{bmatrix} a & \vdots & b \\ \cdots & \vdots & \cdots \\ c & \vdots & d \end{bmatrix}\begin{bmatrix} e & \vdots & f \\ \cdots & \vdots & \cdots \\ g & \vdots & h \end{bmatrix}=\begin{bmatrix} ae+bg & \vdots & af+bh \\ \cdots & \vdots & \cdots \\ ce+dg & \vdots & cf+dh \end{bmatrix}$$

Chapter three // Same type of matrices

1-The transpose of matrix

The transpose, A^T , of an ($m \times n$) matrix A is the ($n \times m$) matrix obtained by interchanging the rows and columns of A, that is, if:

$$A = [a_{ij}]_{m\times n} \quad then$$
$$A^T = A' = [a_{ji}]_{n\times m}$$

Ex: 1- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

2-If $B = [1 \quad 2]$ then $B' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The properties of the transpose of matrix:

Let A and B be two matrices suitable for adding and multiplying, and α be constant then:

1- $(A')' = A$
2- $(\alpha A)' = \alpha A'$
3- $(A+B)' = A' + B'$
4- $(AB)' = B'A'$
$\neq A'B'$

Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and α=4 show that:

1- $(A')' = A$
2- $(\alpha A)' = \alpha A'$

2-The symmetric of matrix:

A square matrix A is said to be symmetric if and only if:

$A' = A$ where $a_{ij}=a_{ji}$ $\forall\, i \; and \; j$

3-Skew symmetric of matrix:(المصفوفة الملتوية التماثل (متماثلة تخالفية

A square matrix A is said to be skew symmetric matrix if and only if:

$A' = -A$ where $a_{ij} = -a_{ji}$ $\forall i \text{ and } j$

Ex: Is this matrix symmetric matrix or Skew symmetric matrix:

1- $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

2- $B = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix}$

3- $D = \begin{bmatrix} -2 & 8 \\ -8 & 0 \end{bmatrix}$

Theorems of the Symmetric and Skew symmetric matrix:

1- For any square matrix A:
- a) $(A + A')' = A' + A$ symmetric.
- b) $(A - A')' = -(A - A')$ skew symmetric.

2- For all matrix A : $A.A' \text{ and } A'.A$ symmetric matrix.

3- If A and B are symmetric matrix then:
- a) $(\alpha A)' = \alpha A$
- b) $(A + B)' = A + B$
- c) If AB =BA then $(AB)' = A.B$

4- If A and B skew symmetric matrix then:
- a) $(\alpha A)' = -\alpha A$
- b) $(A + B)' = -(A + B)$
- c) $(AB)' = A.B$ if A.B=B.A

5- If $A' = A$, $B' = -B$ and A.B=B.A then A.B skew symmetric matrix.

The complex numbers الاعداد المركبة (المعقدة) ژمارەی ئاوێتە :

If **a** , b are real number, then a+ib is named **complex number ,** where **(** $i = \sqrt{-1}$ **).** Then **a** is named ***real part(الجزء الحقيقي)*** of complex number and **ib** is named ***imaginary part (الجزء الخيالي)*** of complex number.

حقيقي خيالي

Z= a + ib

Ex: Z= 3+ 2i , r = 3i , q = -1+4i

Algebric operation of the complex number العمليات الجبرية للاعداد المركبة

Let $Z_1 = a + bi$

$Z_2 = c + di$

1) **Addition of the two complex number جمع الاعداد المركبة :**
$Z_1 + Z_2 = (a + bi) + (c + di)$

$= (a + c) + (bi + di)$

$= (a + c) + (b + d)i$

2) **Subtraction ofthe two complex number طرح الاعداد المركبة :**
$Z_1 - Z_2 = (a + bi) - (c + di)$

$= (a - c) + (bi - di)$

$= (a - c) + (b - d)i$

3) **Multiplication of the two complex number ضرب الاعداد المركبة :**
$Z_1 . Z_2 = (a + bi) . (c + di)$

$= (a.c - bd) + (cb + a d)i$

Ex: Let $Z_1 = 4 + 3i$ and $Z_2 = 5i - 2$ find

1) $Z_1 + Z_2$ 2) $Z_1 - Z_2$ 3) $Z_1 . Z_2$

4) **Multipl the complex number by real number ضرب الاعداد المركبة بعدد حقيقى :**
Let Z= a + bi and *k* is real number then:

$k.Z = k(a+bi)$

$= ka + kbi$

Ex: Let Z = 3 +6i and k=3 find k.Z

Complex matrix المصفوفة المعقدة

It is the matrix that contain element in the complex number.

Ex: $A = \begin{bmatrix} 4+5i & 3-2i \\ 1+2i & 5-3i \end{bmatrix}$

The conjugate of a matrixمرافقة المصفوفة

A matrix $A=((a_{ij}))$ of order (m×n) its named the conjugate matrix and symbolized by $\overline{A}=((\overline{a}_{ij}))$ of order (m×n).

Ex: 1) Let $A=\begin{bmatrix}4+5i & 3-2i\\ 1+2i & 5-3i\end{bmatrix}$ then the conjugate of A is

$$\overline{A}=\overline{\begin{bmatrix}4+5i & 3-2i\\ 1+2i & 5-3i\end{bmatrix}}=\begin{bmatrix}4-5i & 3+2i\\ 1-2i & 5+3i\end{bmatrix}$$

2) Let $B=\begin{bmatrix}3 & 2-3i\\ 1-4i & 3i\end{bmatrix}$ then the conjugate of B is

The tranjugate of matrix:(مرافقة لمبدلة المصفوفة)مبدلة المرافقة

A matrix $A=((a_{ij}))$ of order (m×n) its named the tranjugate of A and symbolized by A^*,

$$A^*=(\overline{A})^T=(\overline{A})'$$
$$=\overline{(A')}$$

Ex: If $A=\begin{bmatrix}2-3i & 3+i\\ 2+2i & 2-i\end{bmatrix}$ find A^*.

The properties of tranjugate of matrix:

1- $(A^*)^*=A$
2- $(kA)^*=k\ A^*$
3- $(A+B)^*=A^*+B^*$
4- $(A.B)^*=B^*.A^*$